AMS 527, S'08, HW 1

1. Compute the condition number $K(d)$ to

$$x - a^d = 0, \qquad a > 0,$$

where d represents "data", a is a fixed parameter, and x is the dependent variable.

2. Consider the Cauchy problem

$$\begin{cases} x\prime(t) = x_0 e^{at}(a\cos(t) - \sin(t)), & t > 0 \\ x(0) = x_0. \end{cases}$$

Determine the large time conditioning of the solution to this ODE with respect to the initial data x_0.

3. The bisection algorithm has the property that each iteration of the algorithm improves the accuracy of the answer by 1 binary digit. How many interations of the bisection algorithm does it take to gain 1 decimal degree of accuracy?

4. With $a_0 = 0$ and $b_0 = 1$, each of the following three functions has the property that $f(a_0)f(b_0) < 0$. For each function, what point does the bisection algorithm locate? Is the point a root of $f(x)$?

a) $f(x) = (3x - 1)^{-1}$

b) $f(x) = \cos(10x)$

c) $f(x) = \begin{cases} 1 & x \geq 0.3 \\ -1 & x < 0.3 \end{cases}$

5. Consider the root-finding problem for the function $f(x) = e^{-x} - \cos x$. Determine an iteration function $\phi(x)$ and an interval $[a, b]$ so that conditions 1 and 3 of Theorem 6.1 hold. (Assume the smallest positive root is to be found.)

6. Let $f(x)$ be a continuous function that is m-times differentiable such that $f(\alpha) = \ldots = f^{(m-1)}(\alpha) = 0$ and $f^{(m)}(\alpha) \neq 0$. Prove that $\phi_N'(\alpha) = 1 - 1/m$, where ϕ_N is the Newton fixed point iteration function. Show that the modified Newton's method

$$x_{k+1} = x_k - m\frac{f(x_k)}{f'(x_k)}$$

restores quadratic convergence for the Newton method.
Hint: Consider Taylor-series expansion with exact remainder for $f(x)$ near α.

7. Consider the fixed point method

$$x_{k+1} = x_k - \frac{f_k}{\phi(x_k)}, \qquad \phi(x_k) = \frac{f(x_k + f_k) - f_k}{f_k}$$

where $f_k = f(x_k)$. Show that this fixed point method is second-order convergent.

8. Consider the fixed point iteration scheme $g(x) = x(2 - ax)$. For any fixed value of a (including the possbility $a = 0$) determine the interval(s) where the scheme converges, and the points to which it converges? As a result of your analysis, determine why this scheme might be useful.

9. How do you modify Descarte's rule of signs to determine the number of negative real roots? Using Descarte's ruls and other properties of roots of polynomials given in the text, determine as much as possible about the location and type of zeros of the polynomial

$$p(x) = x^4 - x^3 + x^2 - x + 1.$$

10. Consider the problem of finding the root of $f(x) = \tan x - x$ which is closest to 50 using the secant method and nine-decimal-digit floating point arithmetic. Is it reasonable to use the termination criterion $|f(x_n)| \leq 10^{-8}$?

11. Assume the fixed-point iteration scheme satisfies $e_{n+1} = k^\alpha e_n$ for some constant k, $|k| < 1$ and $\alpha > 0$. Find an expression for the number of iterations N required to reduce the initial error $e_0 = x_0 - x^*$ by a factor of 10^{-m}, $(m > 0)$.