AMS 527, S'08, HW 2

1. Consider the following polynomial, written in Newton form (via Horner's rule) with centers $z_1, \ldots, z_n$,

$$p(x) = a_0 + (x - z_1)(a_1 + \ldots + (x - z_n)(a_n) \ldots) \, .$$

Show that, if $z_1 = z_2 = \ldots = z_{m+1}$, then

$$p^{(j)}(z_1) = j! a_j, \quad j = 0, \ldots m.$$

2. Let $p_n(x) \in \Pi_n$. Show that the Taylor series expansion of p_n at the point z is just the Newton form of p_n with center z having multiplicity n.

3. Write the Taylor series expansion

$$\sin^{-1}(x) = \sum_{odd \ n} \frac{1 \cdot 3 \cdots (n-2)}{2 \cdot 4 \cdots (n-1)} \, \frac{x^n}{n}$$

in a form that is efficient for evaluation.

4. Let $p(x)$ be a polynomial of degree n.

a) Show that the divided difference $p[x_0, x_1, \ldots, x_n]$ is independent of the interpolation points $x_0, x_1, \ldots, x_n$.

b) Show that the $n + 1$'st divided difference is 0.

5. Let $p_n(x)$ interpolate $f(x)$ at the points $x_0 < x_1 < \ldots < x_n$.

a) For $n = 2$, show that $p_2'(x)$ is the straight line that has the value $f[x_i, x_{i+1}]$ at the point $(x_i + x_{i+1})/2$.

b) Generalize the result in (a) to shown that, for general n, $p_n'(x)$ is the polynomial interpolant to the data $(\xi_i, f[x_i, x_{i+1}])$, for appropriate ξ_i in $[x_i, x_{i+1}]$.

6. Consider the arc of a circle written in parameteric form

$$P(t) = (\sin(t), \cos(t)), \quad 0 \le t \le \pi/2.$$

a) Give the explicit form for the second order parametric interpolation $p_2^s(t), p_2^c(t)$ on the equally spaced values $0 = t_0 < t_1 < t_2 = \pi/2$.

b) What is the largest value of the difference

$$\left| \sin^2(t) + \cos^2(t) - (p_2^s(t))^2 - (p_2^c(t))^2 \right|$$

on $[0, \pi/2]$?

7. Let $g(x) = f[x_0, \ldots, x_n, x]$.

a) Show $g''(x) = 2f[x_0, \ldots, x_n, x, x, x]$.

b) By induction, show $g^{(n)}(x) = n!f[x_0, \ldots, x_n, x, \ldots, x]$. How many times is x repeated on the RHS of this formula?

8. Given the values $y, f_y, f'_y, z, f_z, f'_z$, find the values of the coefficents a_0, a_1, a_2, a_3 for the cubic polynomial

$$p_3(x) = a_0 + a_1(x - y) + a_2(x - y)^2 + a_3(x - y)^3,$$

such that

$$p_3(y) = f_y, \quad p'_3(y) = f'_y, \quad p_3(z) = f_z, \quad p'_3(z) = f'_z.$$