1. Solution. Let $F(x,a) = a_1 e^{a_2 x}$. Assume data are $(x_i, y_i), \quad i = 1, 2, ..., m$ and minimize the distance-squared. Let

$$D(a_1, a_2) = \sum_{i=1}^{m} (y_i - a_1 e^{a_2 x_i})^2,$$
$$\frac{\partial p}{\partial a_1} = -2 \sum_{i=1}^{m} (y_i - a_1 e^{a_2 x_i}) e^{a_2 x_i} = 0,$$
$$\frac{\partial p}{\partial a_2} = -2 \sum_{i=1}^{m} (y_i - a_1 e^{a_2 x_i})(a_1 e^{a_2 x_i} x_i) = 0.$$

Then we have normal equations

$$\sum_{i=1}^{m} y_i e^{a_2 x_i} = a_1 \sum_{i=1}^{m} y_i e^{2 a_2 x_i}$$
$$\sum_{i=1}^{m} y_i x_i e^{a_2 x_i} = a_1 \sum_{i=1}^{m} x_i e^{2 a_2 x_i}.$$

□

2. Solution.

$$L_{k+1} = \frac{2k+1}{k+1} x L_k(x) - \frac{k}{k+1} L_{k-1}(x),$$
$$L_0(x) = 1, \quad L_1(x) = x,$$
$$L_2(x) = \frac{3}{2}(x^2 - \frac{1}{3}), \quad Ł_3(x) = \frac{5}{2}(x^3 - \frac{3}{5}x);$$
$$\int_{-1}^{1} L_3(x)(ax^2 + bx + c)dx = \frac{5}{2} a \int_{-1}^{1} (x^3 - \frac{3}{5}x) x^2 dx + b \int_{-1}^{1} (x^3 - \frac{3}{5}x) x dx + c \int_{-1}^{1} L_3(x) dx.$$

Both $L_3(x)$ and $L_3(x)x^2$ are odd functions, and $\int_{-1}^{1}(x^3 - \frac{3}{5}x)xdx = 2\int_0^1 (x^4 - \frac{3}{5}x^2)dx = 0$. Therefore $\int_{-1}^{1} L_3(x)(ax^2 + bx + c)dx = 0$.

□

3. Solution. The first five Laguerre polynomials for $\alpha = 0$ are

$$< g, h > = \int_0^{+\infty} g(x)h(x)x^0 e^{-x} dx = \int_0^{+\infty} g(x)h(x)e^{-x} dx,$$

where $c_k = -\frac{1}{k+1}, \alpha_k = 2k+1, \beta_k = \frac{k}{k+1}$. $\quad p_{k+1} = -\frac{1}{k+1}(x - (2k+1))p_k(x) - \frac{k}{k+1} p_{k-1}(x)$. Take

$p_0(x) = 1, P_{-1} = 0$. Then

$$\begin{aligned}
p_1(x) &= -(x-1)p_0(x) = -(x-1);\\
p_2(x) &= \frac{1}{2}(x^2 - 4x + 2);\\
p_3(x) &= \frac{1}{6}(-x^3 + 9x^2 - 18x + 6);\\
p_4(x) &= \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24);\\
p_5(x) &= \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 200).
\end{aligned}$$

□

4. Proof. By induction,

(i) $k = 0, p_0(x) = 1, \quad \int_{-1}^{1} dx = 2 =< p_0, p_0 >$, the statement is true.

(ii) We assume that $k = m-1, \int_{-1}^{1}[p_{m-1}(x)]^2 dx = \frac{2}{2(m-1)+1}$.

Then $k = m$, we show $< p_m, p_m >= \frac{2}{2m+1}$. We use 3-term recurrence relation for Legendre polynomials, obtaining

$$\begin{aligned}
p_{m+1} &= \frac{(2m+1)xp_m - mp_{m-1}}{m+1},\\
< p_m, p_m > &= \frac{2m-1}{m} < p_m, xp_{m-1} > - \frac{m-1}{m} < p_m, p_{m-2} >\\
&= \frac{2m-1}{m} < p_m, xp_{m-1} >;\\
< p_{m+1}, p_{m-1} > &= \frac{< (2m+1)xp_m, p_{m-1} >}{m+1} - \frac{m}{m+1} < p_{m-1}, p_{m-1} >= 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
< p_m, xp_{m-1} > &= \frac{m, < p_{m-1}, p_{m-1} >}{2m+1} = \frac{2m}{(2m+1)(2m-1)},\\
< p_m, p_m > &= \frac{2m-1}{m}\frac{2m}{(2m+1)(2m-1)} = \frac{2}{2m+1}.
\end{aligned}$$

□

5. Proof. Let

$$\begin{aligned}
l_i(x) &= \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_k)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_k)},\\
l_j(x) &= \frac{(x-x_0)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_k)}{(x_j-x_0)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_k)}.
\end{aligned}$$

For $\forall i \neq j, l_i(x)l_j(x) = (x-x_0)(x-x_1)\cdots(x-x_k)g(x).x_0, x_1, \cdots, x_k$ are $k+1$ distinct roots of $p_{k+1}(x)$. Let $p_{k+1}(x) = a_{k+1}(x-x_0)(x-x_1)\cdots(x-x_k), l_i(x)l_j(x) = \frac{p_{k+1}(x)g(x)}{a_{k+1}}$, $p_0, p_1, \cdots, p_{k+1}$ are orthogonal, $< l_i(x), l_j(x) >= \int_a^b l_i(x)l_j(x)dx = \int_a^b \frac{p_{k+1}(x)g(x)}{a_{k+1}} = 0$.

□

6. Solution.

$$p(x) = \sum_{k=0}^{2} \frac{< f, p_k >}{< p_k, p_k >} p_k(x),$$
$$f(x) = \sin \pi x.$$

$$k = 0, \frac{< f, p_0 >}{< p_0, p_0 >} = \frac{1}{2} \int_{-1}^{1} sin\pi x dx = 0;$$
$$k = 1, \frac{< f, p_1 >}{< p_1, p_1 >} = \frac{3}{2} \int_{-1}^{1} x sin\pi x dx = \frac{3}{\pi};$$
$$k = 2, \frac{< f, p_2 >}{< p_2, p_2 >} = \frac{5}{2} \int_{-1}^{1} \frac{3}{2}(x^2 - \frac{1}{3}) sin\pi x dx = 0;$$
$$p(x) = \frac{3}{\pi} p_1(x) = \frac{3}{\pi} x$$

Therefore 5 place accuracy $\frac{3}{\pi} = 0.95493$.

□

7. Solution. Let $y = 2x - 1$, then $f(x) = \frac{sin\pi x}{[x(1-x)]^{\frac{3}{2}}} = \frac{sin\pi \frac{(y+1)}{2}}{[\frac{(y+1)}{2}\frac{(1-y)}{2}]^{\frac{3}{2}}} = \frac{8 sin\frac{\pi}{2}(y+1)}{[(y+1)(1-y)]^{\frac{3}{2}}}$.
$I = \int_0^1 f(x)dx = 4\int_{-1}^{1} \frac{sin\frac{\pi}{2}(y+1)}{[(y+1)(1-y)]^{\frac{3}{2}}} dy$.
Let $g(y) = \frac{sin\frac{\pi}{2}(y+1)}{(y+1)(1-y)}$. We notice L'Hospital rule,

$$\lim_{y \to 1} g(y) = \lim_{y \to 1} \frac{\frac{\pi}{2} cos\frac{\pi}{2}(y+1)}{-2y} = \frac{\pi}{4},$$
$$\lim_{y \to -1} g(y) = \lim_{y \to -1} \frac{\frac{\pi}{2} cos\frac{\pi}{2}(y+1)}{-2y} = \frac{\pi}{4},$$

The singularity of $g(y)$ at $y = \pm 1$ can be cancelled. We have $I = 4\int_{-1}^{1} \frac{g(y)}{(1-y^2)^{\frac{1}{2}}} dy$. By using chebyshev-based Gauss Quad,

$$I \approx \frac{4\pi}{k+1} \sum_{i=0}^{k} g(\xi_i),$$

where $\xi_i = cos(\frac{(2i+1)\frac{\pi}{2}}{k+1}), i = 0, 1, ..., k$. Let $k = 999, I \approx 11.1791$.

□