AMS 310: Survey of Probability and Statistics
Sample Final Exam

Last Name: ________________ **First Name:** ________________ **ID:** ________________

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1. Suppose that there are 5 fish in the market and 2 of them are rotten.

   (a) If you select three fish at random, what is the probability that two of them are rotten?

   $$\frac{\binom{3}{1}\binom{2}{2}}{\binom{5}{3}} = \frac{3}{10}$$

   (b) If you select three fish at random, what is the expected number of rotten fish you get?

   $$3 \cdot \frac{2}{5} = \frac{6}{5}$$

2. Let X be a discrete random variable taking values in $\{0, 3, 6\}$ with probability mass function $P(X = 0) = \frac{c}{2}, P(X = 3) = \frac{c}{3}, P(X = 6) = \frac{c}{6}$, where c is a constant.

   (a) Find c and the variance $Var(X)$ of X.

   $$\frac{c}{2} + \frac{c}{3} + \frac{c}{6} = 1 \Rightarrow c = 1$$

   $$E[X] = 3 \cdot \frac{1}{3} + 6 \cdot \frac{1}{6} = 2 \qquad E[X^2] = 9 \cdot \frac{1}{3} + 36 \cdot \frac{1}{6} = 9$$

   $$Var(X) = E[X^2] - E^2[X] = 5$$

   (b) Calculate the probability $P(X \le 2)$.

   $$P(X \le 2) = P(X = 0) = \frac{1}{2}$$

3. Jack and Jill went up a hill to fetch a pail of water. There was a 50% chance for Jack to fall down on the return trip. If Jack fell down there was a 50% chance that Jill would fall down after him. Otherwise, Jill would not fall down. Given that Jill did not fall down, what is the probability that Jack did not fall down?

   $E = \{\text{Jack falls down}\}$ $\quad F = \{\text{Jill falls down}\}$

   $P(E) = 0.5, \quad P(F|E) = 0.5, \quad P(F|E^c) = 0$

   $$P(E^c|F^c) = \frac{P(F^c|E^c) \cdot P(E^c)}{P(F^c|E^c) \cdot P(E^c) + P(F^c|E) \cdot P(E)} = \frac{0.5 \cdot 1}{0.5 \cdot 1 + 0.5 \cdot 0.5} = \frac{2}{3}$$

4. Suppose that X is a continuous random variable which is exponentially distributed with parameter $\lambda = 2$. Define $Y = e^{-2X}$.

(a) Find $P(X \geq 5)$.

$$f(x) = 2e^{-2x}$$

$$P(X \geq 5) = e^{-2 \cdot 5} = e^{-10}$$

(b) Find $P(Y \leq 3)$.

$$P(Y \leq 3) = P(e^{-2X} \leq 3) = P(X \geq -\tfrac{1}{2} \ln 3) = 1$$

5. Suppose that X is a continuous random variable with probability density function

$$f(x) = \begin{cases} c/x^3 & \text{if } x \geq 1, \\ 0 & \text{if } x < 1. \end{cases}$$

where c is some constant. Find the probability $P(X \geq 3)$.

$$1 = \int_1^{\infty} \frac{c}{x^3}\, dx = \left[-\frac{c}{2x^2}\right]\Big|_1^{\infty} = \frac{c}{2}$$

$$\Rightarrow c = 2$$

$$P(X \geq 3) = \int_3^{\infty} \frac{2}{x^3}\, dx = \frac{1}{9}$$

6. Suppose that we observe a random variable X having a normal distribution with mean μ and variance σ^2. A random sample X_1, X_2, X_3, X_4 of size 4 is taken, where each X_i indicates the number of successes out of n trials in the ith observation process, $i = 1, 2, 3, 4$.

(a) Is the estimator $\theta_1 = \frac{X_1+X_2+X_3}{3}$ an unbiased estimator for estimating μ? What about the estimator $\theta_2 = \frac{2X_1+2X_2-X_3+X_4}{4}$?

$$E[\theta_1] = E\left[\frac{X_1+X_2+X_3}{3}\right] = \frac{1}{3}\mu + \frac{1}{3}\mu + \frac{1}{3}\mu = \mu$$

$$E[\theta_2] = E\left[\frac{1}{2}X_1 + \frac{1}{2}X_2 - \frac{1}{4}X_3 + \frac{1}{4}X_4\right] = \mu$$

both unbiased.

(b) Which estimator is more efficient?

$Var(\theta_1) = \frac{1}{3}\sigma^2$ $\quad Var(\theta_2) = \frac{5}{8}\sigma^2$

θ_1 is more efficient since $Var(\theta_1) < Var(\theta_2)$

7. The weights of salmon grown at a fish farm are normally distributed with a standard deviation of 1.2 pounds. Suppose a random sample of 16 fish yielded an average weight of 7.2 pounds. Construct a 95% confidence interval for estimating the true mean weight.

$\alpha = 0.05$, $z_{\alpha/2} = z_{0.025} = 1.96$

A 95% C.I. is

$$\left[7.2 - 1.96 \cdot \frac{1.2}{\sqrt{16}},\ 7.2 + 1.96 \cdot \frac{1.2}{\sqrt{16}}\right]$$

or. $[6.61,\ 7.79]$

8. A consumer protection agency wants to test a paint manufacturer's claim that the average drying time of his new fast-drying paint is 20 minutes. 36 cans of such paint were tested with the intention of rejecting the claim if the mean of the drying times $\bar{X}$ exceeds 20.5 minutes. Otherwise, it will accept the claim.

(a) find the probability of a type I error;

$$P(\bar{X} > 20.5 \mid \mu = 20) = 1 - F\left(\frac{20.5 - 20}{2.4/\sqrt{36}}\right) = 1 - F(1.25)$$

$$= 0.1056$$

(b) find the probability of a type II error when $\mu = 21$ minutes

$$P(\bar{X} < 20.5 \mid \mu = 21)$$

$$= F\left(\frac{20.5 - 21}{2.4/\sqrt{36}}\right) = F(-1.25) = 0.1056$$

9. A random sample of 6 steel beans has a mean compressive strength of 58392 pounds per square (psi) with a standard deviation of 648 psi. Use this information and the level of significance $\alpha = 0.05$ to test whether the true average compressive strength of the steel from which this sample came is 58000 psi. Assume normality.

$H_0: \mu = 58000$ v.s. $H_1: \mu \neq 58{,}000$

Level of significance $\alpha = 0.05$

use t-statistics $t = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}}$.

Two sided test, rejection region $t < -t_{0.025}$ or $t > t_{0.025}$

$t = \dfrac{58392 - 58000}{648/\sqrt{6}} = 1.48 < t_{0.025} = 2.571$, do not reject H_0

10. There are two road from A to B and two roads from B to C. Each of the four roads is blocked by snow with probability 0.2, independent of the others. Find the probability that there is at least an open road from A to C.

P(open road from A to C)

$=P$(open road from A to B $\cap$ open road from B to C)

$=[1 - P$(both roads from A to B are blocked)$] \times$

$[1 - P$(both roads from B to C are blocked)$]$

$= [1 - 0.2^2]^2 = 0.9216$