

AMS 310: Survey of Probability and Statistics  
Sample Test

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1. Suppose we have a continuous probability distribution

$$f(x) = \begin{cases} c(x+4) & \text{if } -4 < x < 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of  $c$ .

(b) Find the cumulative distribution function  $F(x)$ .

②  $\int_{-4}^0 c(x+4) dx = c \cdot \left[ \frac{x^2}{2} + 4x \right] \Big|_{-4}^0 = c(-8+16) = 8c \Rightarrow c = \frac{1}{8}$

③ if  $x \leq -4$   $F(x) = 0$

if  $-4 < x < 0$

$$F(x) = \int_{-4}^x \frac{1}{8}(y+4) dy = \frac{1}{8} \left[ \frac{y^2}{2} + 4y \right] \Big|_{-4}^x = \frac{1}{8} \left[ \frac{x^2}{2} + 4x + 8 \right]$$

if  $x \geq 0$

$$F(x) = 1$$

$$F(x) = \begin{cases} 0 & \text{if } x \leq -4 \\ \frac{1}{8} \left[ \frac{x^2}{2} + 4x + 8 \right] & \text{if } -4 < x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

2. Let  $X$  and  $Y$  be two discrete random variables with joint mass function

$$f(x, y) = \begin{cases} c(3x+2y) & \text{if } x = 1, 2, 3, y = 1, 2, 3, \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of  $c$ .

(b) Find the marginal distributions  $f_X(x)$  and  $f_Y(y)$ .

(c) Are  $X$  and  $Y$  independent? Justify your answer.

④  $\sum_{\text{all } x} \sum_{\text{all } y} f(x, y) = c(3+2) + c(3+4) + c(3+6) + c(6+2) + c(6+4) + c(6+6) + c(9+2) + c(9+4) + c(9+6) = 90c = 1 \Rightarrow c = \frac{1}{90}$

⑤  $f_X(x) = \sum_{\text{all } y} f(x, y) = \sum_{\text{all } y} \frac{1}{90} (3x+2y) = \frac{1}{90} (3x+2) + \frac{1}{90} (3x+4) + \frac{1}{90} (3x+6) = \frac{1}{90} (9x+12) = \frac{x}{10} + \frac{2}{15} \quad x=1, 2, 3$

$$f_Y(y) = \sum_{\text{all } x} f(x, y) = \frac{1}{90} (3+2y) + \frac{1}{90} (6+2y) + \frac{1}{90} (9+2y) = \frac{1}{15} y + \frac{1}{5}$$

⑥ No.

Because  $f(x, y) \neq f_X(x) \cdot f_Y(y)$

3. Let  $X$  be a continuous random variable having the probability density function

$$f(x) = \begin{cases} 2(1-x) & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the mean  $\mu$  and variance  $\sigma^2$  of  $X$ .

(b) Determine the median  $Q_2$ , the third quartile  $Q_3$ , and the 92th percentile of  $X$ .

$$\textcircled{a} \quad \mu = \int_0^1 x f(x) dx = \int_0^1 2x(1-x) dx = \frac{1}{3}$$

$$E[X^2] = \int_0^1 x^2 f(x) dx = \int_0^1 2x^2(1-x) dx = \frac{1}{6}$$

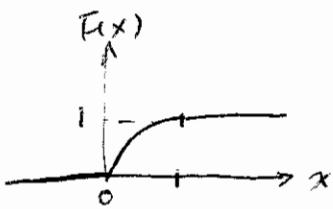
$$\sigma^2 = E[X^2] - \mu^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

$$\textcircled{b} \quad F(x) = \int_0^x 2(1-y) dy = 2x - x^2 \quad \text{if } 0 \leq x \leq 1$$

$$Q_2: F(x) = 2x - x^2 = \frac{1}{2} \Rightarrow x = 1 - \frac{\sqrt{2}}{2}$$

$$Q_3: F(x) = 2x - x^2 = \frac{3}{4} \Rightarrow x = \frac{1}{2} \quad \text{circled } x = \frac{3}{2}$$

$$92\text{th}: F(x) = 2x - x^2 = \frac{92}{100} \quad \text{circled } x = 1.28 \quad x = 0.7172$$



4. Let  $W$  be a continuous random variable with exponential distribution with parameter  $\lambda = 2$ .

(a) Find the mean  $\mu$  and variance  $\sigma^2$  of  $W$ .

(b) Prove that  $P(W > 5|W > 2) = P(W > 3)$ .

(c) Determine  $P(W > 4)$ .

$$f(w) = \lambda e^{-\lambda w} = 2e^{-2w} \Rightarrow F(w) = 1 - e^{-2w} \quad w > 0$$

$$\textcircled{a} \quad \mu = \frac{1}{2} \quad \sigma^2 = \frac{1}{4}$$

$$\textcircled{b} \quad P(W > 5|W > 2) = \frac{P(W > 5, W > 2)}{P(W > 2)} = \frac{P(W > 5)}{P(W > 2)} = \frac{1 - F(5)}{1 - F(2)} = \frac{e^{-10}}{e^{-4}} = e^{-6}$$

$$P(W > 3) = 1 - F(3) = e^{-6}$$

$$\textcircled{c} \quad P(W > 4) = 1 - F(4) = e^{-8}$$

5. The age  $X$  of a typical subscriber to a certain newspaper is a normal random variable with mean  $\mu = 35.5$  years old and standard deviation  $\sigma = 4.8$  years. What is the probability that  $X$  is between 30 and 40 years old?

$$X \sim N(35.5, 4.8^2)$$

$$P(30 \leq X \leq 40) = P\left(\frac{30 - 35.5}{4.8} \leq \frac{X - 35.5}{4.8} \leq \frac{40 - 35.5}{4.8}\right)$$

$$= P(-1.1458 \leq Z \leq 0.9375)$$

$$= F(0.9375) - F(-1.1458)$$

$$= 0.8264 - 0.1251$$

$$= 0.7013$$

6. Let  $X$  and  $Y$  be two continuous random variables with joint density function

$$f(x, y) = \begin{cases} \frac{4}{9}(2+xy) & \text{if } 0 < x < 1, 0 < y < 1. \\ 0 & \text{otherwise} \end{cases}$$

Find  $P(X > \frac{1}{2}|Y > \frac{1}{2})$ .

$$P(X > \frac{1}{2}|Y > \frac{1}{2}) = \frac{P(X > \frac{1}{2}, Y > \frac{1}{2})}{P(Y > \frac{1}{2})}$$

$$P(X > \frac{1}{2}, Y > \frac{1}{2}) = \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \frac{4}{9}(2+xy) dx dy = \int_{\frac{1}{2}}^1 \left[ \frac{4}{9} \left[ 2x + \frac{x^2 y}{2} \right] \right]_{\frac{1}{2}}^1 dy = \int_{\frac{1}{2}}^1 \frac{4}{9} \left( \frac{3y}{8} + 1 \right) dy$$

$$f_Y(y) = \int_0^1 \frac{4}{9}(2+xy) dx = \frac{4}{9} \left( 2x + \frac{x^2 y}{2} \right) \Big|_0^1 = \frac{8}{9} + \frac{2}{9}y. \quad = \frac{41}{144}.$$

$$P(Y > \frac{1}{2}) = \int_{\frac{1}{2}}^1 \left( \frac{8}{9} + \frac{2}{9}y \right) dy = \frac{8}{9}y + \frac{y^2}{9} \Big|_{\frac{1}{2}}^1 = \frac{19}{36}$$

$$\Rightarrow P(X > \frac{1}{2}|Y > \frac{1}{2}) = \frac{41}{144} \cdot \frac{19}{36} = \frac{41}{76}$$

7. The mean of a random sample of size  $n = 25$  is used to estimate the mean of an infinite population that has standard deviation  $\sigma = 2.4$ . Find the probability that the error will be less than 1.2.

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_{25}}{25} \quad E[\bar{X}] = \mu \quad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{2.4}{5} = 0.48$$

$$P(|\bar{X} - \mu| \leq 1.2)$$

$$= P(-1.2 \leq \bar{X} - \mu \leq 1.2) = P\left(-\frac{1.2}{0.48} \leq \frac{\bar{X} - \mu}{\sigma_{\bar{X}}} \leq \frac{1.2}{0.48}\right)$$

$$= P(-2.5 \leq \frac{\bar{X} - \mu}{\sigma_{\bar{X}}} \leq 2.5)$$

$$= F(2.5) - F(-2.5) = 0.9938 - 0.0062$$

$$= 0.9876$$

8. Let  $X_1, X_2, \dots, X_{25}$  be a random sample drawn from a normal population with (unknown) mean  $\mu$  and standard deviation 0.04. What is the probability distribution of the sample mean  $\bar{X}$ ? Find the probability that the sample mean  $\bar{X}$  will deviate from the true mean  $\mu$  by at most 0.02?

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_{25}}{25} \sim N(\mu, (\frac{0.04}{5})^2)$$

$$P(|\bar{X} - \mu| \leq 0.02)$$

$$= P(-0.02 \leq \bar{X} - \mu \leq 0.02)$$

$$= P\left(-\frac{0.02}{0.008} \leq \frac{\bar{X} - \mu}{\sigma_{\bar{X}}} \leq \frac{0.02}{0.008}\right) = P(-2.5 \leq Z \leq 2.5)$$

$$= 0.9876.$$