EULERIAN FRONT TRACKING FOR SOLID DYNAMICS

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ABSTRACT. An attractive approach for simulation of solid dynamics is to combine an Eulerian finite difference method with material interface tracking. The fixed Eulerian computational mesh is not subject to mesh distortion, and the tracking eliminates spurious numerical diffusion at interfaces and the need for mixed-material computational cells. We have developed such an approach within the framework of the front tracking method, as implemented in the Frontier code. Our two-dimensional solid dynamics code is based on a fully conservative formulation of the governing equations for large-strain deformation, a hyperelastic equation of state that allows for large volumetric change, and a rate-dependent plasticity model for high strain rates; it features conservative finite differencing, a Riemann solver that accounts for the nonlinearity of longitudinal waves, and an implicit method for integrating the plastic source term. This paper presents an overview of the Frontier-Solid code and some preliminary applications to high-velocity impact and shock-accelerated interface problems.

1. INTRODUCTION

The simulation of dynamic material failure and fragmentation, such as occurs during ballistic impact loading [27], is an important challenge for multi-scale science. From this point of view, individual microstructural features (e.g., grains, ductile voids, brittle cracks, and adiabatic shear bands) are the elementary modes whose collective evolution at the mesoscale leads to macroscale dynamic failure. Through study of these elementary modes, both by analysis and simulation, multi-scale methods effect a coarse-graining to the macroscale. This approach ultimately yields continuum models, suitable for use in system-level simulation codes, which incorporate the integrated effects of the evolving material microstructure. Direct numerical simulation, in which relevant microstructural features are explicitly represented in the computational domain, is critical to the success of multi-scale modeling for dynamic failure in solids. Moreover, because such simulations are used in the construction
of macroscale models, a premium must be placed on high resolution numerical methods and physical fidelity of the underlying constitutive descriptions.

An attractive approach for simulation of solid dynamics is to combine an Eulerian finite difference method with material interface tracking. The fixed Eulerian computational mesh is not subject to mesh distortion, and the tracking eliminates spurious numerical diffusion at interfaces and the need for mixed-material computational cells. Moreover, microscale descriptions of solid failure surfaces are accurately coupled to the surrounding media by the solution of non-local Riemann problems along the tracked fronts. Front tracking, as implemented in the FrOnTier code (see, e.g., Ref. [10]), is particularly well suited to this approach.

A conservative Eulerian formulation of the equations of motion for elasto-visco-plastic materials was first proposed by Plohr and Sharp [16, 17]. This formulation allows the well-developed technology of conservative finite difference methods to be applied to solid dynamics. Fully conservative Godunov methods for one-dimensional elastodynamics have been studied by several authors [1, 6], who note improvement in solution quality. Others [22] have developed a Godunov scheme for one-dimensional rate-independent plastic flow; however, their scheme is only partially conservative, being based on a non-conservative stress evolution equation. Wang et al. [25] developed and compared fully conservative Eulerian Godunov and Lax-Wendroff schemes for rate-dependent plasticity in a uniaxial geometry; these authors reported good agreement with plate impact experiments.

We have developed an Eulerian front tracking code for solid dynamics. The current implementation is applicable to plane strain flow of ductile metals. This code is based on a fully conservative formulation of the governing equations for large-strain deformation, a hyperelastic equation of state that allows for large volumetric change, and a rate-dependent plasticity model for high strain rates; it features conservative finite differencing, a Riemann solver that accounts for the nonlinearity of longitudinal waves, and an implicit method for integrating the plastic source term. A linearized version of the the Riemann solver is currently used to implement the front propagation algorithm for metal/vacuum and frictionless metal/metal material interfaces. This will be replaced by the nonlinear solver in the near future.

This paper provides an overview of the FrOnTier-Soli\textsuperscript{d} code and progress in simulating of impact problems and shock-accelerated interfaces. The content of the paper is as follows. In the next section, we briefly recount the conservative Eulerian formulation of the governing equations. In the Sec. 3, we explain some of the details of our numerical method. Finally, in Sec. 4, we present some results from numerical simulations of high-velocity impact and shock-accelerated interface problems.
2. **Conservative Eulerian Formulation**

In the conservative Eulerian formulation [17], the state of a deformed material body is characterized by the inverse deformation gradient, the particle velocity, a single thermodynamic variable, and several internal variables that account for plasticity. The motion of the body is determined by conservation principles.

2.1. **Deformation.** Consider the finite deformation of a body, first from the Lagrangian point of view. Let \( X^\alpha, \alpha = 1, 2, 3 \) denote the material coordinates for the (undeformed) Lagrangian configuration, and let \( x^i, i = 1, 2, 3 \) denote the spatial coordinates for the (current, deformed) Eulerian configuration. The motion of the material is specified by a time-dependent map \( \phi^i \) specifying the Eulerian coordinates \( x^i \) of each material point \( X^\alpha \):

\[
x^i = \phi^i(X^\alpha, t). \tag{2.1}
\]

The gradient of \( \phi^i \) is the matrix \( F^i_\alpha := \partial \phi^i / \partial X^\alpha \), which is called the deformation gradient, and the time derivative of \( \phi^i \) (with \( X^\alpha \) fixed) is the particle velocity \( V^i := \dot{\phi}^i \). Equating mixed partial derivatives yields the conservation law \( \dot{F}^i_\alpha = \partial V^i / \partial X^\alpha \), which represents the continuity of the body.

The corresponding Eulerian equations are based upon the inverse of the relationship (2.1):

\[
X^\alpha = \psi^\alpha(x^i, t). \tag{2.2}
\]

The gradient of \( \psi^\alpha \), denoted \( g^\alpha_i := \partial \psi^\alpha / \partial x^i \), is called the inverse deformation gradient because when \( x^j \) and \( X^\beta \) are corresponding points, \( g^\alpha_i(x^j, t) \) is the matrix inverse of \( F^i_\alpha(X^\beta, t) \). Taking the time derivative of the identity \( X^\alpha = \psi^\alpha(\phi^j(X^\beta, t), t) \), we find that the time derivative of \( \psi^\alpha \) is \( \partial \psi^\alpha / \partial t = -g^\alpha_k v^k \), where \( v^i(x^j, t) := V^i(X^\beta, t) \) when \( x^j \) and \( X^\beta \) are corresponding points. Again equating mixed partial derivatives [22] yields the conservation law

\[
\frac{\partial g^\alpha_i}{\partial t} + \frac{\partial (g^\alpha_j v^j)}{\partial x^i} = 0. \tag{2.3}
\]

If \( g^\alpha_i \) and \( v^i \) solve this equation, and if, at \( t = 0 \), \( g^\alpha_i \) is the gradient of a map \( \psi_0^\alpha \), then there exists an inverse motion \( \psi^\alpha \), equaling \( \psi_0^\alpha \) at \( t = 0 \), from which \( g^\alpha_i \) and \( v^i \) derive for \( t \geq 0 \). Therefore, by including Eq. (2.3) among the conservation laws, we can regard \( g^\alpha_i \) and \( v^i \) as fundamental dynamical variables, instead of \( \psi^\alpha \), and thereby reduce to a first-order system. See Refs. [16, 23] for further discussion.

**Remark.** In this paper we treat only plane strain. Therefore the components \( g^3_1, g^3_2, g^1_3 \), and \( g^2_3 \) vanish, and \( g^3_3 = 1 \).

2.2. **Hyperelasticity.** Deformation leads to stresses within the body. We assume that the material is hyperelastic, in that its response to deformation and thermodynamic change is determined by the functional dependence of the specific internal energy \( \varepsilon \) on the inverse
deformation gradient \( g^a_i \), the entropy \( \eta \), and various internal variables. For example, if the material is a gas, \( \varepsilon \) depends on \( g^a_i \) only through the mass density

\[
\rho := \rho_0 \det g
\]

(2.4)

(where \( \rho_0 \) is the mass density when the body is undeformed), and the derivative of \( \varepsilon \) with respect to the specific volume \( \tau := \rho^{-1} \) is the mean stress \(-p\), where \( p \) is the hydrodynamic pressure. In general, the energy depends only on changes in the distances between points of the body; it is unaffected by spatial rotations. Therefore the energy depends on the deformation gradient only through the strain tensor \( E_{\alpha\beta} \), as defined by

\[
E := \frac{1}{2} (F^T F - I).
\]

(2.5)

(Here and in the following we regard \( F^i_a \) as a function of the spatial position \( x^j \) rather than the material position \( X^\beta \).) In the small-strain limit, \( E_{\alpha\beta} \) reduces to the infinitesimal strain \( \varepsilon_{ij} \), i.e., the symmetric part of the displacement gradient.

2.3. Plasticity. Plastic materials exhibit permanent deformation. To model this phenomenon [9], we introduce the components of a symmetric “plastic” strain tensor \( E_{\alpha\beta} \), analogous to \( E_{\alpha\beta} \), as internal variables. Also, to model strain hardening, we use an additional internal variable, the equivalent plastic strain \( \kappa \). Thus the energy \( \varepsilon \) depends on \( E_{\alpha\beta} \) and \( \kappa \) as well as \( E_{\alpha\beta} \) and \( \eta \).

Because we are modeling ductile metals, we shall assume that the material response is isotropic. In this context, the form of the dependence of \( \varepsilon \) on \( E_{\alpha\beta} \) and \( E_{\alpha\beta} \) can be motivated by a microscopic picture that suggests a multiplicative decomposition of the deformation gradient into elastic and plastic parts [4, 12, 13]. The result is that the energy depends solely on three independent isotropic invariants of the tensor \((I + 2 E\delta)^{-1}(I + 2 E)\). (In the small-strain limit, this tensor reduces to the \( \delta_{ij} + 2 \varepsilon_{ij}^e \), where \( \varepsilon_{ij}^e := \varepsilon_{ij} - \varepsilon_{ij}^p \) is the infinitesimal elastic strain.) See Refs. [17, 15] for further discussion.

Remark. In keeping with standard models of ductile metal plasticity, we assume that plastic yielding is volume preserving, i.e., \( \det(I + 2 E\delta) = 1 \); therefore, in plane strain, the plastic strain tensor has only three independent components, viz., \( E_{11}^p \), \( E_{12}^p \), and \( E_{22}^p \).

The isotropic invariants of \((I + 2 E\delta)^{-1}(I + 2 E)\) coincide with those of the elastic Finger tensor \( \tilde{b}^i_j \), defined by

\[
\tilde{b}_e := F(I + 2 E\delta)^{-1} F^T.
\]

(2.6)

A particular choice of three invariants achieves a clean separation between volumetric and shear strains: \( J := \sqrt[3]{\det \tilde{b}_e} \) (or, equivalently, \( \rho = \rho_0 / J \) or \( \tau = J / \rho_0 \)) and two independent invariants of the the reduced elastic Finger tensor \( \tilde{b}^i_j \), defined by

\[
\tilde{b}_e := J^{-2/3} \tilde{b}_e,
\]

(2.7)
which is insensitive to volumetric change. In the context of plane strain, there is only one
independent invariant of the reduced elastic Finger tensor, which we take to be
\[ \epsilon^2 := \frac{1}{2} (\text{tr} \, b_\nu - 3). \] (2.8)
The invariant \( \epsilon \) measures the elastic shear strain. (In the small-strain limit, \( \epsilon \) reduces to the
norm of the deviator of \( \epsilon_{ij}^e \).)

2.4. Governing equations. The dynamics of the body is governed by the conservation
law (2.3) along with conservation of momentum, conservation of energy, and the evolution
equations for the internal variables.

\[
\begin{align*}
\frac{\partial}{\partial t} (g^a_i) + \frac{\partial}{\partial x^j} (g^a_{k^j}) &= 0, \\
\frac{\partial}{\partial t} (\rho v^i) + \frac{\partial}{\partial x^j} (\rho v^i v^j - \sigma^{ij}) &= 0, \\
\frac{\partial}{\partial t} (\rho e) + \frac{\partial}{\partial x^j} (\rho e v^j - \sigma^{ij} v_i) &= 0, \\
\frac{\partial}{\partial t} (\rho E^p_{\alpha\beta}) + \frac{\partial}{\partial x^k} (\rho E^p_{\alpha\beta} v^k) &= \rho \Lambda_{\alpha\beta}, \\
\frac{\partial}{\partial t} (\rho \kappa) + \frac{\partial}{\partial x^k} (\rho \kappa v^k) &= \rho h.
\end{align*}
\] (2.9) (2.10) (2.11) (2.12) (2.13)

In these equations \( \sigma^{ij} \) is the Cauchy stress, \( \epsilon := \frac{1}{2} v_i v^i + \epsilon \) is the total energy per unit mass,
\( \Lambda_{\alpha\beta} \) is the plastic source term, and \( h \) is the hardening source term.

2.5. Constitutive equations. To complete the governing system, we require constitutive
equations for \( \sigma^{ij} \) and \( \epsilon \) and a plastic flow rule that defines \( \Lambda_{\alpha\beta} \) and \( h \). Since we assume that the
response of the material to deformation is hyperelastic, an equation of state for \( \epsilon \) also
determines \( \sigma^{ij} \).

We employ a model for the specific internal energy \( \epsilon \) founded on the assumption that the
elastic shear strain \( \epsilon \) remains small:

\[ \epsilon := \epsilon_{\text{hydro}} (\tau, \eta) + \tau G (\tau, \eta) \epsilon^2 + O(\epsilon^3). \] (2.14)

In making this choice we have combined ideas from several authors [24, 8, 19, 7, 18]. The first
term represents the hydrostatic contribution to the energy and allows for large volumetric
changes; the second term, in which \( G \) is the shear modulus, accounts for small elastic shear
strain. For this internal energy, the Cauchy stress \( \sigma^{ij} \) can be calculated to be

\[
\sigma^{ij} := \rho F^i_\alpha \frac{\partial \epsilon}{\partial E_{\alpha\beta}} F^{ij}_\beta = -p \delta^{ij} + G (\text{dev} \, b_\nu)^{ij} + O(\epsilon^2),
\] (2.15)

where \( p := -\partial \epsilon_{\text{hydro}} / \partial \tau \) is the hydrostatic pressure. (We use the notation \( \text{dev} \, A \) for the
deviator of a \( 3 \times 3 \) matrix \( A \), defined by \( \text{dev} \, A := A - \frac{1}{3} (\text{tr} \, A) I \).) In the implementation
of the numerical code, we neglect the higher-order terms in Eqs. (2.14) and (2.15).

Remark. In plane strain, \( \sigma^{13} \) and \( \sigma^{23} \) vanish.
For $\varepsilon_{\text{hydro}}$ we use a stiffened polytropic equation of state (see, e.g., Ref. [14]):

$$
\varepsilon_{\text{hydro}}(\tau, \eta) := \frac{(p_0 + p_\infty)}{\Gamma \rho_0} (\rho_0 \tau)^{-\Gamma} \exp[\Gamma \eta/R] + p_\infty \tau,
$$

where $p_0$, $p_\infty$, $\Gamma$, and $R$ are material constants. The shear modulus $G$ is taken from the work of Steinberg et al. [20]:

$$
G(\tau, \eta) := G_0 \left[ 1 + G_p (\rho_0 \tau)^{1/3} p + G_T (T - T_0) \right],
$$

where $G_0$, $G_p$, and $G_T$ are also material constants. Here the hydrostatic pressure $p$ and temperature $T$ are calculated as first derivatives of the hydrostatic part of the internal energy:

$$
p(\tau, \eta) := -\frac{\partial \varepsilon_{\text{hydro}}}{\partial \tau}(\tau, \eta) = (p_0 + p_\infty)(\rho_0 \tau)^{-(\Gamma + 1)} \exp[\Gamma \eta/R] - p_\infty,
$$

$$
T(\tau, \eta) := \frac{\partial \varepsilon_{\text{hydro}}}{\partial \eta}(\tau, \eta) = T_0 (\rho_0 \tau)^{-\Gamma} \exp[\Gamma \eta/R],
$$

where $T_0 := (p_0 + p_\infty)/(\rho_0 R)$.

2.6. Flow rule. The plastic and hardening source terms are determined by the flow rule. These source terms are nonzero, i.e., the material undergoes plastic flow, only if the shear stress exceeds a certain threshold. We adopt the rate-dependent plasticity model of Steinberg and Lund [21], as modified by Wang et al. [25].

Let

$$
Y := \sqrt{\frac{3}{2}} \| \text{dev} \sigma \|
$$

denote the dynamic yield strength. (The norm $\|A\|$ of a matrix $A$ is defined by $\|A\|^2 = \text{tr} (A^T A)$.) The material undergoes plastic flow only if $Y$ exceeds the static yield strength, i.e.,

$$
Y > \frac{G(\tau, \eta)}{G_0} Y_A(\kappa);
$$

otherwise $h$ and $\Lambda_{\alpha\beta}$ vanish and the material response is elastic. Here

$$
Y_A(\kappa) := \min \{ Y_0 (1 + \beta \kappa)^n, Y_{\text{max}} \},
$$

is the athermal part of the dynamic yield strength, and $Y_0$, $\beta$, $n$, and $Y_{\text{max}}$ are material constants. The dependence of the static yield strength on the equivalent plastic strain $\kappa$ models strain hardening.

When $Y$ exceeds the static yield strength, the thermal part $Y_T > 0$ of the dynamic yield strength is defined through the decomposition

$$
Y = \frac{G(\tau, \eta)}{G_0} [Y_A + Y_T].
$$
Then the plastic and hardening source terms $\Lambda_{\alpha\beta}$ and $h$ are given by

\[
\Lambda_{\alpha\beta} := \sqrt{\frac{3}{2}} \left\| \text{dev} \sigma \right\| \left[ F^T (\text{dev} \sigma) b^1_b F \right]_{\alpha\beta}, \tag{2.24}
\]

\[
h := \frac{Y_T}{C_2 + \frac{Y_T}{C_1} \exp \left[ \text{sgn} (Y_P - Y_T) \frac{2U_k}{kT} \left( 1 - \frac{Y_T}{Y_P} \right)^2 \right]}, \tag{2.25}
\]

where $C_1$, $C_2$, $U_k/k$, and $Y_P$ are more material constants. For more details about the flow rule, see Refs. [17, 15, 25].

3. **Numerical Method**

Equations (2.9)–(2.13) constitute a system of conservation laws with source terms,

\[
W_t + F(W)_x + G(W)_y = H(W), \tag{3.1}
\]

for the state vector $W = (g^a_i, \rho v^i, \rho e, \rho E^b_{\alpha\beta}, \rho k)^T$. Indeed, all other quantities appearing in Eqs. (2.9)–(2.13), such as $\rho$, $v^i$, $\sigma^{ij}$, $\Lambda_{\alpha\beta}$, and $h$, are determined uniquely from $W$. A description of this calculation appears below in Sec. 3.2.

3.1. **Summary of the algorithm.** The numerical code we have developed for solving system (3.1) is based on the front tracking method, as implemented in the *FrontTier* code. In this method, the discrete solution is represented by state values on two-dimensional grid and along one-dimensional tracked curves. These curves represent material interfaces and boundaries. Enhanced accuracy of the numerical solution is achieved by avoiding interpolation across curves.

The state values and curve positions are updated using an explicit time stepping as follows. First the tracked curves are moved, and their associated states are updated, using the front propagation algorithm described in Secs. 3.5 and 3.8. Then for each point on the two-dimensional grid, a regular nine-point stencil, centered on this point, is constructed and the numerical scheme described in Secs. 3.3 and 3.4 is applied. If the nine-point stencil is crossed by a tracked curve, the states on the stencil are obtained by interpolation between states within the same connected component as the center point. To each grid cell is associated physical and geometrical information (e.g., lists of components and bonds [3, 10]) for facilitating efficient calculation of nearest interface points. Grid cells crossed by the front are triangulated, and this triangulation is used for interpolation of states.

For further details of the implementation of the *FrontTier* code, refer to Refs. [2, 10]; and for the history and summary of front tracking methods, refer to Ref. [11].

3.2. **Recovery algorithm.** All required quantities associated with a state can be recovered from the state vector $W$ using the following procedure.

1. Calculate $\rho = \rho_0 \det g$; then obtain $v^i$, $e$, and $E_{\alpha\beta}$.
2. After finding $\tilde{b}_b$ from Eqs. (2.6) and (2.7), determine $e^2$ using Eq. (2.8).
3. Find $\eta$ from Eq. (2.14) combined with Eqs. (2.16)-(2.19) (taking advantage of how $\exp[\Gamma \eta/R]$ appears linearly); then calculate $p$, $T$, and $G$ from Eqs. (2.18), (2.19), and (2.17).

3. From the variables already obtained, find $\sigma^{ij}$ using Eq. (2.15).

4. From the value of $\sigma^{ij}$ obtained, evaluate $Y$ using Eq. (2.20) and find $Y_T$ from Eq. (2.23).

5. Finally, find $\Lambda_{\alpha \beta}$ and $h$ from Eqs. (2.24) and (2.25).

3.3. Interior scheme. In the interior (i.e., off of the tracked material interfaces), system Eq. (3.1) is solved using a spatially unsplit, two-step Lax-Wendroff method [5] and using temporal splitting for the source term. The solution is represented by cell-centered values on a regular rectangular grid with spacings $\Delta x$ and $\Delta y$; spatial indexing is denoted by a pair of subscripts, whereas a superscript indicates the time step.

The treatment of the plastic source terms is described in Sec. 3.4. The scheme for advancing by a time increment $\Delta t$, ignoring the source terms, involves the following two steps:

1. Half step on the dual grid:

$$W_{i+1/2,j+1/2} = \hat{W}_n^{i+1/2,j+1/2} + \frac{\Delta t}{2\Delta x} (\hat{F}_n^{i+1/2,j+1/2} - \hat{F}_n^{i,j+1/2}) + \frac{\Delta t}{2\Delta y} (\hat{G}_n^{i+1/2,j+1} - \hat{G}_n^{i+1/2,j+1/2}),$$

(3.2)

$$\hat{W}_n^{i+1/2,j+1} = \frac{1}{4}(\hat{W}_n^{i+1,j+1} + \hat{W}_n^{i+1,j} + \hat{W}_n^{i,j+1} + \hat{W}_n^{i,j}),$$

(3.3)

$$\hat{F}_n^{i+1/2,j+1} = \frac{1}{2}[F(W_n^{i+1,j+1}) + F(W_n^{i+1,j})],$$

(3.4)

$$\hat{G}_n^{i+1/2,j+1} = \frac{1}{2}[G(W_n^{i+1,j+1}) + G(W_n^{i,j+1})].$$

(3.5)

2. Full step on the primal grid:

$$W_{i,j}^{n+1} = W_{i,j}^n + \frac{\Delta t}{\Delta x} (\hat{F}_{i+1/2,j}^{n+1/2} - \hat{F}_{i-1/2,j}^{n+1/2}) + \frac{\Delta t}{\Delta y} (\hat{G}_{i,j+1}^{n+1/2} - \hat{G}_{i,j-1}^{n+1/2}),$$

(3.6)

$$\hat{F}_{i+1/2,j}^{n+1/2} = \frac{1}{2}[F(W_{i+1/2,j+1}^{n+1/2}) + F(W_{i+1/2,j-1}^{n+1/2})],$$

(3.7)

$$\hat{G}_{i,j+1}^{n+1/2} = \frac{1}{2}[G(W_{i+1/2,j+1}^{n+1/2}) + G(W_{i+1/2,j}^{n+1/2})].$$

(3.8)

In addition, linear artificial viscosity is used to control oscillations in the solution caused by large-strength interior (i.e., untracked) waves.

3.4. Plastic update. The plastic source term, $H(W)$, in Eq. (3.1) is integrated with a second-order Runge-Kutta step followed by an implicit loop to ensure convergence. Only the equations for $\rho E_{\alpha \beta}^p$ and $\rho \kappa$ have possibly nonzero source terms; the remaining conserved quantities, which constitute the elastic part of the state, are frozen during the plastic update.

For brevity we describe only the update of the equivalent plastic strain, $\kappa$, for which the source term is given by Eqs. (2.20), (2.22), (2.23), and (2.25). The first and second
Runga-Kutta substeps and the implicit loop are given by

\[
\bar{r}^{n+1/2} = \bar{r}^n + \frac{\Delta t}{2} h(\bar{r}^n),
\]

(3.9)

\[
\kappa_0^{n+1} = \bar{r}^n + \Delta t h(\bar{r}^{n+1/2}),
\]

(3.10)

\[
\kappa_{\alpha+1}^{n+1} = \bar{r}^n + \frac{\Delta t}{2} [h(\kappa_{\alpha+1}^{n+1}) + h(\bar{r}^n)].
\]

(3.11)

Here \( \bar{r}^n \) has already been updated using Eqs. (3.2)–(3.8), and \( \alpha \) labels the iterate in the implicit loop.

An estimate of the maximum allowable plastic timestep \( \Delta t_p \) is obtained from the convergence criterion for Eq. (3.11) which is

\[
\frac{1}{2} |\frac{\partial h}{\partial \kappa}| \Delta t_p < C_f,
\]

(3.12)

where \( C_f = \delta^{1/N} < 1 \) is the convergence factor, \( \delta \) is the relative error tolerance, and \( N \) is the desired number of iterations.

From the analysis of uniaxial flow in Ref. [15], it follows that

\[
\frac{\partial Y_T}{\partial \epsilon_p} \approx 2G, \quad \frac{\partial \epsilon_p}{\partial \kappa} \approx 3/2,
\]

(3.13)

where \( \epsilon_p \) is the uniaxial plastic strain, so that \( \Delta t_p \) is restricted by

\[
\Delta t_p < \frac{2C_f}{3G |\partial h/\partial Y_T|}.
\]

(3.14)

For the modified Steinberg-Lund model in the (linear) viscous drag regime (viz., where \( Y_T \) exceeds the Peierls stress, \( Y_P \)), we have that \( h(Y_T) \approx Y_T/C_2 \), so that (3.14) may be written as

\[
\Delta t_p < 2C_f \tau_p
\]

(3.15)

where \( \tau_p := C_2/(3G) \) is the plastic relaxation time. For tantalum, \( C_2 = 3.5 \) GPa-\( \mu s \) and \( G = 69 \) GPa [26], so that \( \tau_p = 0.017 \) \( \mu s \) in the drag regime. At lower impact speeds \( (Y_T < Y_P) \), the same analysis applies, but the dependence of \( h \) on \( Y_T \) is more complicated, with the minimum value for \( \tau_p \) being about half of its value in the drag regime. An estimate of the hyperbolic grid spacings \( \Delta x \) and \( \Delta y \) required to spatially resolve plastic waves may be obtained by multiplying the elastic longitudinal wave speed by \( \Delta t_p \). An error of \( 10^{-3} \) achieved after two iterations corresponds to \( \Delta x \approx 5 \mu m \).

In practice, the iterative loop is run with a target iteration count of two. If convergence is obtained after one, two, or three iterations, then an acceptable \( \Delta t_p \) for the next step is set, respectively, to 1.2, 1, or 0.75 times the current \( \Delta t \). If convergence is not obtained, \( \Delta t \) is halved and the time step is attempted anew.
3.5. **Front propagation.** A distinctive component of the front tracking method is the front propagation algorithm, and the basis of this algorithm is the solution of Riemann problems, i.e., one-dimensional initial-value problems containing jump discontinuities.

At each of its points, the front provides a natural splitting of the flow into normal and tangential directions. A front point has two associated states, corresponding to the two sides of the front. As the first step in front propagation algorithm, these states are taken as the left and right states for a Riemann problem aligned along the normal direction. Solving this Riemann problem determines, to a first approximation, the normal propagation speed of the front. Next, the influence of states near to, but not on, the front are taken into account using a combination of the method of characteristics and the Rankine-Hugoniot conditions, which provides corrections to the propagation speed and to the states on the front. These two steps solve a nonlocal Riemann problem in the normal direction. To complete the front propagation algorithm, waves are propagated tangentially along the front using a one-dimensional finite difference scheme.

3.6. **Riemann solver for material with shear strength.** Efficient numerical schemes for solving Riemann problems are well-known for gas dynamics; at the core of such a scheme is the solution of a single nonlinear equation. In contrast, solving a Riemann problem for a material with shear strength poses a significant challenge: it involves solving a large system of nonlinear equations, and the convergence of methods for solving this system is difficult to guarantee.

We have developed an efficient Riemann solver for tracking the interface between two solid materials that are slipping past each other without friction. (A specialization of this solver applies to the interface between a solid material and a gas.) Two features of the frictionless interface problem allow for simplification: (1) certain conditions at the interface (the vanishing of the tangential tractions) can be solved explicitly, thereby reducing the size of the system of nonlinear equations; and (2) shear waves are weak and therefore can be treated as linear. After taking advantage of these features, the core of the scheme reduces to solving a single nonlinear equation, the evaluation of which involves two single-equation fixed-point iterations. The outer nonlinear equation has a form similar to that for gas dynamics, and the inner iterations correspond to a standard equation of state calculation. Because of its simple structure, the Riemann solver is efficient and robust.

As the details of the implementation of this Riemann solver require an explanation that is too lengthy for the present paper, we defer a thorough discussion to a future paper.

3.7. **Demonstration of the Riemann solver.** As a simple demonstration of the Riemann solver, results are shown for a series of Riemann problems corresponding to head-on collision of tantalum bodies. (See Sec. 4.1 for the material parameters used.)

In the first series of Riemann problems, the material is initially in the natural, undistorted state (i.e., \( g^0 = I \) is the identity matrix, and \( p, E_{a,b}, \) and \( \kappa \) are zero); the relative velocity is \( 2v \). The solution of such a Riemann problem contains two longitudinal shock waves that compress the material in the middle. The solid curve in Fig. 3.1 is a plot of the compressive
normal traction as the velocity \( v \) is varied from 0.5 to 5 mm/\( \mu \)s. The curvature of this plot reflects the nonlinearity of the equation of state.

The second series of Riemann problems is the same except that the material is initially deformed: the \( g^1_2 \) component of the deformation gradient is set to 0.4. The results from these Riemann problems are plotted as the dashed curve in Fig. 3.1. The effect of this large initial shear is slight.

3.8. **Triple node.** Solid deformation problems with complex geometry will typically contain triple points at which discontinuity curves intersect. In front tracking, special algorithms are needed for such points, called nodes. In the present case, three tracked material interfaces intersect at a **triple node**. In an impact problem, triple nodes occur at the ends of the line of contact between the target and penetrator, the third material being void (or low density gas). Propagation of a node is idealized as a localized two-dimensional Riemann problem in which the curves are approximated by straight lines separating wedge-shaped regions. The analytic solution of such a two-dimensional Riemann problem is only known in some special cases. For the triple node between very dissimilar materials (as with a void and two solid materials), the idealized solution appears to require that one of the three sectors defined by the three curves has a zero opening angle. Our algorithm is build on this picture. Here we briefly describe the algorithm for this node propagation.

A triple node is the meeting point of one slip curve \( (C_2) \) and two approximately parallel incident curves \( (C_0 \) and \( C_1 \), as drawn in Fig. 3.2. We assume that the two incident curves
Figure 3.2. Sketch of the triple node. The curves $C_0$ and $C_1$ are the two approximately parallel incident curves, and $C_2$ is the slip curve.

are “closing up,” so that the slip surface is lengthening. The propagation of triple node occurs by two mechanisms. In the first, the opening angle between the incident curves is reduced, but the triple node does not move in material coordinates. In the second, the triple node moves discretely in material coordinates.

Let us define $P_j$ as the point adjacent to node at the opposite end of the bond at the node on curve $j$. First, the current node position is updated by applying the front normal point propagate algorithm to the slip curve $C_2$ alone. A stencil is assembled using states from the non-void sides of incident curves and both sides of slip curve; the void sector is ignored. The front propagation is also performed along the curves $C_0$ and $C_1$ now with one on-front void state at $(P_0, P_1)$, but only to update state data at the node. Second, propagation of other points on the incident curves tends to narrow the void sector angle at the node. After several steps a cross will occur, perhaps several bonds away from the node. When this occurs the node is simply relocated to the position of the cross. Relocating the node in this way also widens the void sector angle. Over several time steps, the processes of narrowing, cross, and relocate can repeat.

The first mechanism operates every step and produces small changes in the node position. The second mechanism occurs less frequently but produces much larger changes in position. In effect, the penetrator and target roll together.
Eventually, as a penetration tunnel begins to form in the target, a reversed flow starts to occur in the penetrator. To accommodate this possibility, the entire triple node undergoes a large (approximately 90 degree) rotation, after which the void sector (between the two incident curves) begins to open steadily towards 180 degrees while the penetrator sector closes toward zero. The target sector (which was never narrower than 120 to 150 degrees) also widens towards 180 degrees. Once the void sector begins widening, the second propagation mechanism ceases and the node is carried along on the surface of the penetration tunnel in an essentially passive manner.

4. Numerical Examples

In this section we present some numerical results obtained using our implementation of the *FronTier-Solid* code.

4.1. Impact problem. We have applied our code to high-velocity impact and penetration problems. As an example, we present results for the impact of a circular projectile disk on a moderately thin plate. Both the disk and the target are modeled as tantalum, for which material parameters are as follows [25, 26]: \( \rho_0 = 16.69 \text{ g/cm}^3 \), \( p_0 = 0 \), \( p_{\infty} = 72.7 \text{ GPa} \), \( \Gamma = 1.67 \), \( T_0 = 300 \text{ K} \), \( G_0 = 69 \text{ GPa} \), \( G_p = 0.0145 \text{ GPa}^{-1} \), \( G_T = -0.13 \cdot 10^{-3} \text{ K}^{-1} \), \( Y_0 = 0.375 \text{ GPa} \), \( Y_{\max} = 0.45 \text{ GPa} \), \( \beta = 22 \), \( n = 0.283 \), \( C_1 = 0.71 \mu s^{-1} \), \( C_2 = 3.5 \text{ GPa} \cdot \mu s \), \( U_k/k = 3600 \text{ K} \), \( Y_f = 0.82 \text{ GPa} \). The impact velocity is 2.5 mm/\( \mu s \) (i.e., 2.5 km/s).

Material interfaces at two different times are shown in Figs. 4.1–4.2. At \( t = 0.03 \mu s \) a pair of shock waves have been established in the plate and penetrator. Referring to Fig. 4.1 the wave in the plate has reflected as a rarefaction from the rear surface causing outward deformation. The wave in the disk has refracted around the moving triple points at the end of the disk-plate contact line but has not yet reached the far edge of the disk. The triple point (of void, target, and disk) requires special algorithmic consideration as discussed above in Sec. 3.8. The behavior of the rate dependent plasticity model is shown in the plot of equivalent stress, \( \sigma_{eq} (= Y) \). Large values occur just behind the the triple nodes where the plate and, to a lesser extent the disk, is suddenly deformed by the advancing contact. As expected, \( \sigma_{eq} \) decays rapidly farther behind the triple node. Stretching of the plate produces peak values of \( \sigma_{eq} \) about one half those near the triple nodes.

At \( t = 0.075 \mu s \) in Fig. 4.2 the flow speed \(|v|\) in the disk and the center of the plate have equilibrated while away from the vertical axis the plate flow speed is significantly smaller due to the frictionless contact condition. So plate material continues to stretch and accumulate values of plastic strain \( (\kappa) \) larger than those in nearby regions of the disk. Also note that a small lip of plate material near the triple node has formed and flow reversal in the penetrator has begun. Thus this computation captures a number of important physical features in a penetration problem.

4.2. Shock-accelerated interface. Of considerable interest is the problem of an unstable material interface which, for fluids, has been the subject of long study. Whenever a heavier material is driven against lighter, the flow is potentially unstable. Here, we consider the
Figure 4.1. Tantalum disk impacting tantalum plate at 2.5 mm/μs. The initial disk radius is 100 mm and the plate thickness in 25 mm. The computational mesh is 100 × 100 square cells 0.005 mm on the edge. Pressure is shaded on the left and flow stress $Y$ (von Mises equivalent stress) on right.

Figure 4.2. Tantalum disk impacting tantalum plate at 2.5 mm/μs. Flow speed $|v|$ shaded on the left and equivalent plastic strain $\kappa$ on the right.

Impulsively driven or Richtmyer-Meshkov instability for tantalum using the material model described herein, with the material parameter values noted previously.

With reference to Fig. 4.3 the velocity, $v^y$, is shaded over the computational domain of 0.5 × 2.0 mm which is covered by a mesh of 100 × 400 square grid cells. At the initial time,
$t = 0$, the solid fills the lower portion of the domain and is in a natural, undistorted state (i.e., $g^a_i$ is the identity matrix, and $p$, $E_{\alpha\beta}$, and $\kappa$ are zero) with $v^x = 0$ and $v^y = -3$ mm/$\mu$s. The metal-vacuum interface is perturbed by two periods of a sine wave so that the initial interface thickness is 0.10 mm with aspect ratio 0.4; the mean height of the interface is 1.2 mm above the bottom of the domain. The left and right boundaries are periodic, the top is flow through and the bottom is reflecting. The negative initial velocity generates a strong shock which propagates upward to load the interface. Conditions behind the incident shock are $\sigma^{yy} = 360$ GPa, $Y = 140$ GPa, $\rho = 28$ g/cm$^3$, and $\kappa = 0.33$.

Immediately after interaction with the shock at 0.172 $\mu$s (second panel) the interface has been compressed but retains a roughly sinusoidal shape. At 0.24 $\mu$s inversion of the interface has begun and by 0.38 $\mu$s the incipient spikes have attained about equal height with the neighboring bubbles. Figs. 4.4–4.5 show that although the incident shock is strong and the perturbation aspect ratio is large, the residual stresses near the interface following interaction with the shock are much smaller. Referring to Fig. 4.5, as expected, there is some plastic strain concentration in the developing spikes. The right pair of plots show the evolution of the free surface and $v^y$ for exactly the same problem but for a stiffened polytropic gas rather than the elastic visco-plastic solid. The gas EOS Eq. (2.16) and material parameters agree with those used for the hydrostatic part of the tantalum internal energy. Thus, because the residual stresses are small, material strength greatly suppresses the speed with which the spikes form in spite of the strong incident shock.

5. Summary

We have developed an Eulerian front tracking method for the high strain-rate deformation of ductile metals, modeled as hyperelastic-viscoplastic materials, and we have successfully applied it to high-velocity impact and shock-accelerated interface problems. We conclude that the algorithms discussed here and the implementation we have produced are able to handle computationally difficult problems, in a convergent and stable manner, with simulation results of an expected nature. In particular, due to front tracking, the material boundaries have the perfect resolution characteristic of a Lagrangian method, although the primary computational mesh is Eulerian. Thus the anticipated advantage of combining Eulerian and Lagrangian methods has been achieved. Shock waves are fully resolved by the interior scheme (without tracking), and are not sharply localized, because of the effect of plasticity, which provides them with a finite width. Thus tracking of the shock waves, which we have not done here, would be scientifically incorrect, unless the simulation were to be attempted on a much coarser grid than that used here. Further refinement of the Frontier-Solid code and application of it to more complex flow and failure simulations are the subject of on-going research.
Figure 4.3. Evolution of a shock driven, sinusoidally perturbed free surface in tantalum. Shading represents the vertical component of material velocity $v^y$ (in mm/µs) at the indicated times (in µs). The initial shape of the free surface (solid below) appears at 0.18 µs just prior to arrival of the shock from below. The initial velocity is $v^y = -3$ mm/µs against a hard lower boundary which generates the shock. Note the separate velocity scales for various plots.

References

Figure 4.4. Vertical normal stress $\sigma_{yy}$ and equivalent stress $\sigma_{eq} = Y$ (in GPa) for the problem in Fig. 4.3. Plots are at the same times as the third and fourth panels in that figure.
Figure 4.5. Left two panels: evolution of equivalent plastic strain, $\kappa$, for the problem in Fig. 4.3. Right two panels: evolution of free surface and $v^y$ (in mm/$\mu$s) for the same problem but with a stiffened polytropic gas. The gas EOS Eq. (2.16) and material parameters agree with those used for the hydrostatic part of the tantalum internal energy.


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