THE EXISTENCE, REGULARITY, AND BEHAVIOR AT INFINITY OF ISOTROPIC
SOLUTIONS OF CLASSICAL GAUGE FIELD THEORIES*

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Abstract

We study isotropic, finite action solutions of certain classical gauge field theories, namely the Abelian-Higgs model in two Euclidean dimensions and the Yang-Mills Higgs model in three Euclidean dimensions. We prove the existence of vortex solutions with arbitrary vortex number and monopole solutions with arbitrary isospin in these respective models, verify that the solutions are smooth everywhere, and give a detailed description of the asymptotic behavior of the solutions at infinity. In particular we prove that the (classical) Higgs phenomenon takes place and we precisely determine the masses of the gauge and Higgs fields; an interesting outcome is that the mass of the Higgs field cannot exceed twice the mass of the gauge field. Our proofs employ variational methods from non-linear functional analysis and techniques from the theory of ordinary differential equations.
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I. Introduction

The theory of the fundamental interactions in nature has undergone an important unification in recent years: the strong, weak, and electromagnetic forces all seem to be best described by quantized gauge field theories. The Weinberg-Salam model ([W1] and [S]) gives a unified description of the weak and electromagnetic interactions of leptons and hadrons as a gauge theory, and quantum chromodynamics [MP] describes the strong interactions of hadrons as a gauge theory. Even in Einstein's theory of gravity one of the basic fields (the affine connection) is of the same type as those in gauge field theories, although the dynamics of the gravitational interaction differ from the dynamics of gauge theories. Thus gauge theories hold the promise of a unified picture of the forces in nature.

A gauge theory is characterized by a group of symmetries. Consider a field theory in which the fields transform according to a representation of some Lie group; suppose the interaction of these fields is invariant under the action of this group. Then we may extend this symmetry of the interaction by requiring that the interaction be invariant under local transformations by this group; that is to say we prescribe that the force remain unchanged if the fields undergo a transformation which may vary from point to point in space and time. As shown by Yang and Mills [YM] this prescription leads one to introduce a so-called gauge field which mediates a new interaction between the original fields in the field theory. In this way the forces due to gauge fields are related to symmetries.

Gauge theories have a number of properties that make them the most promising candidates for the theory of fundamental interactions. Because of the Higgs mechanism [H], the particle spectrum of a gauge theory may lack manifest symmetry, as it does in nature, even though gauge theories have such a high degree of symmetry. In certain gauge theories the inter-
action becomes weaker at shorter distances; thus gauge theories can describe inelastic lepton-hadron scattering, which displays this property of asymptotic freedom [P1]. More far reaching is the prospect of viewing the various forces in nature as consequences of a single unifying symmetry principle [W2]. (The reader is referred to Taylor [Ta] for more discussion of the properties of gauge theories.)

In addition to their being important in physics, gauge theories have a rich mathematical structure. In their classical versions they may be interpreted geometrically in terms of fiber bundles. This interpretation brings powerful techniques from differential and algebraic geometry to bear on the problem of classifying classical solutions of the equations of motion. In Appendix I we briefly discuss this geometric picture of gauge fields. In the main body of this thesis, though, we will be content with a more mundane description of gauge fields; we use only some elementary terminology from differential geometry because of its convenience for calculation. (Specifically, we speak of differential forms, the exterior differentiation operator $d$, and the Hodge duality operator $\ast$; these notions are defined, e.g., in [CDD].)

One striking feature of gauge theories is that their classical equations of motion admit non-trivial solutions. These solutions are analogous to the soliton solutions of 2-dimensional scalar field theories; but solitons appear in scalar field theories only in two dimensions, whereas field theories with gauge symmetry have non-trivial solutions in higher dimensions. Many physicists have argued that these solutions play an important role in quantized gauge theories (see [C]); consequently these solutions have been studied extensively.

For example, Nielsen and Olesen [NO] consider a gauge theory, called the Abelian Higgs model, that describes a self-interacting scalar field whose
group of symmetries is the group $U(1)$. They reduce the problem of finding isotropic solutions known as vortices to solving a coupled pair of non-linear ordinary differential equations together with boundary conditions. A similar set of equations is obtained by t'Hooft [t'H] and Polyakov [Py] for isotropic solutions in the Yang-Mills Higgs model, which involves a self-interacting scalar field with symmetry group $SU(2)$. It is the purpose of this thesis to prove that solutions of these boundary-value problems indeed exist, to prove that they give rise to smooth solutions of the equations of motion of gauge theories, and to determine some of their properties. Our proofs employ well-known techniques from non-linear functional analysis and the theory of ordinary differential equations.

In Chapter II we treat the Abelian Higgs model, which we define more precisely in II.1. In II.2 we give an overview of the proofs of existence, regularity, and behavior of solutions; the details of these proofs are given in II.3, II.4, and II.5 respectively. Chapter III is devoted to the Yang-Mills Higgs model, described in III.1. In III.2, III.3, and III.4 we prove the existence, regularity, and behavior of solutions of the Yang-Mills Higgs model using ideas similar to those applied to the Abelian Higgs model. A summary of our results is given in the Conclusion, wherein we discuss further the importance of classical gauge theories.
II.1 The Abelian Higgs Model

The Abelian Higgs model describes a charged scalar Higgs field which is self-coupled via a potential $V$ and which interacts with an Abelian (i.e. U(1)) gauge field in 2-dimensional Euclidean space. Thus the Higgs field $\phi$ is a complex-valued function on $\mathbb{R}^2$, and the gauge field may be written as $iA$, where $A$ is a real-valued 1-form on $\mathbb{R}^2$. (We represent the Lie algebra of U(1) as $i\mathbb{R}$.) The potential $V$ is assumed to be twice continuously differentiable, non-negative, and symmetric about the origin; furthermore we assume that $V$ has a zero at $R_o > 0$ such that $V''(R_o) > 0$ and $V(R) > 0$ if $|R| < R_o$ (see Fig. 1). Using units in which $R_o = 1$ and $e/\hbar c = 1$ (where the charge of the Higgs field is taken, for convenience, to be $-e < 0$), the Euclidean action for the Abelian Higgs model is

$$\mathcal{J}(\phi, A) = \int_{\mathbb{R}^2} d^2x \left\{ \frac{1}{2} \| dA \|^2 + \frac{1}{2} \| d_A \phi \|^2 + V(\| \phi \|) \right\},$$

where $d = d + iA$. The critical points of $\mathcal{J}$ formally satisfy the equations

$$d^* d_A \phi + V'(\| \phi \|) \frac{\phi}{\| \phi \|} = 0 \quad \text{and} \quad d^* d A + \frac{1}{2i} \left( \overline{\phi} d_A \phi - \phi d_A \overline{\phi} \right) = 0;$$

(here $^*$ denotes the formal adjoint with respect to the appropriate inner product, and $\overline{\cdot}$ denotes complex conjugation). These are the Abelian Higgs equations.

We remark that the above equations are also obtained in the Landau-Ginzburg theory [TI] of superconductivity (if one uses units in which
$\hbar^2 R_0^2/m = 1$ and $e/\hbar c = 1$, where $m$ is the electron mass); the function $\mathcal{F}$ is then the Landau-Ginzburg free energy per unit length.

We wish to construct so-called vortex solutions of (II.1.1), characterized by having a non-zero vortex number (i.e. first Chern number)

$$\eta = \frac{i}{2\pi} \int_{\mathbb{R}^2} \mathcal{J}(iA),$$

which is an integer. Following Nielsen and Olesen [NO] we look for isotropic solutions of the form

$$\phi(x,y) = R(r) \exp(i n \theta)$$

and

$$A(x,y) = S(r) \, d\theta,$$

where $n$ is an integer, and $r$ and $\theta$ are polar coordinates defined by $x = r \cos \theta$ and $y = r \sin \theta$. If the real-valued functions $R$ and $S$ satisfy

$$-R''(r) - r^{-1} R'(r) + r^{-2} (S(r) + n)^2 R(r) + V'(R(r)) = 0$$

and

$$-S''(r) + r^{-1} S'(r) + R^2(r) (S(r) + n) = 0,$$

then $\phi$ and $A$, given by (II.1.2), satisfy (II.1.1) away from the origin:

$$d_A \phi(x,y) = \left( d + i S(r) \, d\theta \right) \cdot R(r) \, e^{i n \theta}$$

$$= \left[ R'(r) \, dr + i R \omega(S(r) + n) \, d\theta \right] e^{i n \theta},$$
so that (with * denoting the Hodge duality operator)

\[ \oint A * A \phi(z,y) = - \star^{-1} \star A * \star A \phi(z,y) \]

\[ = - \star^{-1} \left[ (r R''(r) - R'(r)) dr \wedge d\theta + r^{-2}(S(r)+n)^2 R(r) d\theta \wedge dr \right] e^{i\theta} \]

\[ = \left[ - R''(r) - r^{-1} R'(r) + r^{-2}(S(r)+n)^2 R(r) \right] e^{i\theta} \]

\[ = - V'(R(r)) e^{i\theta} = - \left[ \frac{V'(\|a\|)}{\|a\|} \right] e^{i\theta} \]

and

\[ \star A(x,y) = \star^{-1} \star A \left[ S'(r) dr \wedge d\theta \right] = \star^{-1} \star A \left[ r^{-2} S'(r) \right] \]

\[ = \star^{-1} \left[ r^{-1} S''(r) - r^{-2} S'(r) \right] dr = \left[ - S''(r) + r^{-1} S'(r) \right] d\theta \]

\[ = - R''(r) (S(r) + n) = - \left[ \frac{1}{2} \left( \overline{\phi} \star A \phi - \phi \star A \phi \right) \right](x,y). \]

The vortex number is

\[ n = - \frac{1}{2\pi} \lim_{r \to \infty} \oint_{\partial B(0,r)} S(r) d\theta = - \lim_{r \to \infty} S(r) \]

by Stokes' theorem (where \( B(0,r) \) is the disk of radius \( r \) centered at the origin). It therefore suffices to construct solutions with \( n \geq 0 \), for otherwise we may replace \( S \) by \(-S\) and \( n \) by \(-n\); since the case \( n = 0 \) is trivial, we assume that \( n \geq 1 \) below.

Critical points of the function \( F \) defined by \( F(R,S) = (2\pi)^{-1} F(\phi, A) \), i.e.

\[ F(R,S) = \int_0^r dr \left\{ \frac{1}{2} r^{-2} [S'(r)]^2 + \frac{1}{2} [R'(r)]^2 \right. \]

\[ + \frac{1}{2} r^{-2} R''(r) (S(r) + n)^2 + V(R(r)) \left\} \right. \]
formally satisfy (II.1-3) and the boundary conditions

\[ R(r) \to 0 \quad \text{and} \quad S(r) \to 0 \quad \text{as} \quad r \to 0 \]

and

\[ R(r) \to 1 \quad \text{and} \quad S(r) \to -n \quad \text{as} \quad r \to \infty. \]

We will prove the existence of solutions of (II.1-1) by precisely defining \( F \) on a space of pairs \((R,S)\), proving that \( F \) attains its infinimum at some point \((R_{\min}, S_{\min})\) in this space, and verifying that \( R_{\min} \) and \( S_{\min} \) satisfy (II.1-3) and the appropriate boundary conditions.
II.2 Overview of the Proof

Many of the standard approaches to proving the existence of solutions of nonlinear equations employ the following simple strategy. One considers a space of possible solutions and a convenient measure of how far a member of this space is from being a solution. One then constructs a sequence of approximate solutions for which this measure tends to zero, extracts a convergent subsequence of this sequence, and proves that the limit of this subsequence is a solution. For example, techniques based on the contraction mapping principle (and related results such as Newton's method and implicit function theorems), Galerkin approximations, and variational principles all implement this basic strategy, although in quite different ways (see [B]).

To solve our nonlinear elliptic boundary-value problem we will use a variational principle. In the previous section we saw that minima of a certain real-valued, nonlinear function $F$ are (at least formally) solutions of our equations. Therefore we will construct a Hilbert space $H$ of possible solutions on which $F$ is defined and apply the above strategy to demonstrate that $F$ attains its infimum $\inf F$ at some point $u_{\text{min}}$ in $H$. In this case the measure of how far a member $u$ of $H$ is from being a minimum is simply $F(u) - \inf F$, and the natural way to obtain a sequence of approximate minima is to pick some minimizing sequence, i.e. a sequence $j + u_j$ for which $F(u_j) + \inf F$. The difficult parts of our program are the extraction of a convergent subsequence from a minimizing sequence and the proof that the limit of this subsequence minimizes $F$.

Consider the problem of extracting a convergent subsequence. Recalling that compact metric spaces have the Bolzano-Weierstrass property (that any sequence has a convergent subsequence), we might try to prove that any minimizing sequence lies inside a compact subset of $H$; but we cannot expect
this to be true because subsets of \( H \) that are compact in the norm topology are too small. However, the Banach-Alaoglu theorem provides a convenient criterion for compactness in the weak topology on \( H \): closed, norm-bounded subsets of a separable Hilbert space have the Bolzano-Weierstrass property. Therefore if we ask only for a weakly convergent subsequence we need only show that any minimizing sequence is bounded in norm. Now the only control we have over the size of \( u_j \) is through \( F(u_j) \), so to show that \( ||u_j|| \leq \text{const.} \) for all \( j \) it is simplest to verify that \( F(v_j) \to \infty \) whenever \( ||v_j|| \to \infty \).

This property of \( F \) is called coercivity. The function \( F \) that we consider, though, happens not to be coercive, and we must resort to a trick for modifying the minimizing sequence we pick to obtain one which is bounded in norm.

Given such a minimizing sequence we can find a \( u_\infty \) in \( H \) and a sequence \( j + u_j \) in \( H \) such that \( F(u_j) \to \inf F \) and \( u_j + u_\infty \) in the weak topology; \( u_\infty \) is our candidate for \( u_{\min} \).

To show that \( F(u_\infty) = \inf F \) it clearly suffices to show that \( F(v_\infty) \leq \lim_{j \to \infty} F(v_j) \) whenever \( v_j \to v_\infty \) weakly. This property of \( F \) is called (sequential) weak lower semicontinuity. There are two convenient types of weakly lower semicontinuous functions which arise in practice. First, there are functions which are strongly continuous and convex. (These may be seen to be weakly lower semicontinuous since convex sets that are strongly closed are also weakly closed; see [V]). Second, there are functions that factor as \( H \hookrightarrow H' \to \mathbb{R} \), where the embedding \( H \hookrightarrow H' \) is a compact map and the map \( H' \to \mathbb{R} \) is strongly continuous. (These functions are actually (sequentially) weakly continuous because compact maps carry weakly convergent sequences to strongly convergent ones; a version of the Sobolev embedding theorem, Lemma II.3.4, together with the Arzela-Ascoli theorem, will provide a compact embedding). If a function is the sum of functions of these types, it too is weakly lower semicontinuous. Our function is not quite of this form,
but the ideas just described can be adapted to prove that \( F(u_\infty) = \inf F \).

In this way we can solve the variational problem of finding a minimum \( u_\min = u_\infty \) of \( F \). We must now justify the formal identification of minima of \( F \) with solutions of our differential equations. To do this we first show that the derivative \( F'(u_\min) \) of \( F \) at \( u_\min \) exists; it then follows from the minimizing property of \( u_\min \) that \( F'(u_\min) = 0 \). This equation satisfied by \( u_\min \), however, is not actually equivalent to the differential equation whose solutions we seek. Indeed, \( u_\min \) may not be sufficiently differentiable to satisfy the differential equation, even though it satisfies the equation in some "integral" sense. We describe this by saying that \( u_\min \) is a weak solution. In II.4, we prove that weak solutions to our equations are always sufficiently regular to actually satisfy the differential equation.

In our method for establishing the regularity of a weak solution we derive a formula that expresses the solution as the antiderivative of a function that involves the solution itself. Thus if the weak solution is sufficiently regular, \textit{a priori}, to guarantee that this function is continuous (this will follow from the finiteness of \( F(u_\min) \) and a Sobolev embedding theorem), then the formula shows that the solution is in fact continuously differentiable; by iterating this argument we can prove that the weak solution is sufficiently differentiable to satisfy the differential equation we wish to solve. This formula, which is crucial to this argument, follows directly from the equation \( F'(u_\min) = 0 \) satisfied by a weak solution \( u_\min \); indeed it merely expresses the vanishing of the derivative of \( F \) in the direction of the vector \( u_G \), where \( u_G \) is (essentially) a Green function.

We must show not only that a weak solution satisfies the differential equation we wish to solve, but also that it satisfies the appropriate boundary conditions. The space \( H \) can be chosen so that the only weak solu-
tions that belong to \( H \) satisfy the boundary conditions, as we will show in II.4 and II.5.

In our case, where the members of \( H \) are functions on the half-line \( \mathbb{R}_+ = ]0, \infty[ \), there is a boundary condition at the origin and at infinity. The boundary condition at the origin arose because we introduced polar coordinates in order to reduce our partial differential equation to an ordinary one. Therefore we must in fact show that no singularity is introduced by using polar coordinates. It turns out that to show this we need only to show that the solution approaches the correct boundary value sufficiently rapidly. Moreover, the integral formula for the solution, which we used to establish regularity, can be used here to demonstrate that the solution behaves appropriately near the origin. (This is the advantage of our method of proving regularity over the other methods available.) In this manner we can show that the physical fields in which we are interested are smooth throughout space.

That a weak solution satisfies the boundary condition at infinity is closely related to the existence of the topological charge of the solution. In II.5 we use techniques from the theory of ordinary differential equations (essentially the WKB approximation) to show that a solution approaches its boundary value at infinity exponentially fast. The characteristic length of this exponential behavior is also of interest, since it is the classical approximation to the inverse mass of the particle which appears in the quantized field theory. We find that the mass exhibits interesting behavior as we vary the sharpness of the minimum of the Higgs self-interaction potential.

Finally, let us explain some of the terminology and notation used below. We will say that a function on Euclidean space is of class \( C^j \), where \( j \) is a positive integer or infinity, if it is \( j \)-times continuously differentiable. For an open subset \( \Omega \) of a Euclidean space and \( d\mu \) a
measure on $\Omega$ we let $L^2(\Omega, d\mu)$ denote the set of real-valued functions on $\Omega$ that are square-integrable with respect to $d\mu$. We also let $C_b(\Omega)$ denote the set of bounded, continuous functions on $\Omega$, and let $C_c^\infty(\Omega)$ denote the $C^\infty$ functions on $\Omega$ that have compact support. Lastly, we write

$$f(x) = O(g(x))$$

as $x \to x_0$ if there is a constant $\rho$ such that

$$|f(x)| \leq \rho |g(x)|$$

for all $x$ in a neighborhood of $x_0$, and we write

$$f(x) = o(g(x))$$

as $x \to x_0$ if $g$ does not vanish in a deleted neighborhood of $x_0$ and

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$
II.3 The Abelian Higgs Model — Existence of Solutions

In II.1 we have reduced the problem of finding isotropic solutions of the Abelian Higgs model to finding real-valued functions $R$ and $S$ on the half-line $[r+] = 0$, $\infty$ that solve the nonlinear elliptic boundary-value problem

$$
\begin{align*}
- R''(r) - r^{-1} R'(r) + r^{-2} (S(r) + n)^2 R(r) + V'(R(r)) &= 0, \\
- S''(r) + r^{-1} S'(r) + R^2(r) (S(r) + n) &= 0,
\end{align*}
$$

(II.3-1)

$$R(r) \to 0 \quad \text{and} \quad S(r) \to 0 \quad \text{as} \quad r \to 0,$n

(II.3-2)

and

$$R(r) \to 1 \quad \text{and} \quad S(r) \to -n \quad \text{as} \quad r \to \infty.$n

(II.3-3)

Here $n$ is the vortex number, an integer, and $V$ is the Higgs self-interaction potential (see Fig. 1). These equations are formally satisfied by any pair $(R_{\min}, S_{\min})$ which minimizes the function $F$ given by

$$F(R, S) = \int_0^\infty r \, dr \left\{ \frac{1}{2} r^{-2} [S'(r)]^2 + \frac{1}{2} [R'(r)]^2 + \frac{1}{2} r^{-2} R^2(r) (S(r) + n)^2 + V(R(r)) \right\}. \quad (II.3-4)$$

Our procedure in proving the existence of solutions will be to precisely define $F$ as a function on a space $H$ of pairs $(R, S)$ and to demonstrate that $F$ attains its infimum at some point $(R_{\min}, S_{\min})$ in $H$. 
An appropriate choice for the space $H$ is critical to the success of this procedure. The space $H$ must be large enough to be complete and yet not so large as to contain minima for $F$ which do not satisfy the boundary conditions; and the topology on $H$ must be strong enough that $F$ is weakly lower semicontinuous but weak enough that $F$ is coercive. Our choice for $H$ is motivated by the form of $F$ and a bit of hindsight (which is explained below).

To define $H$ we first introduce two functions $\hat{R}$ and $\hat{S}$ that satisfy the boundary conditions required of our solution. Let $R$ be a real-valued $C^\infty$ function on $\mathbb{R}_+$ such that $0 \leq \hat{R}(r) \leq 1$ for all $r \in \mathbb{R}_+$, $\hat{R}(r) = 0$ if $r \leq 1$, and $\hat{R}(r) = 1$ if $r \geq 2$; and let $\hat{S}(r) = -n \hat{R}(r)$ for all $r \in \mathbb{R}_+$ (see Fig. 2). If we write $R = \hat{R} + \bar{R}$ and $S = \hat{S} + \bar{S}$ the boundary conditions for $R$ and $S$ become the conditions that $\bar{R}(r)$ and $\bar{S}(r)$ vanish as $r \to 0$ and $r \to \infty$. We will take $\bar{R}$ to belong to the space $H_R$ which consists of those $\bar{R}$ in $L^2(\mathbb{R}_+, r dr)$ whose distributional derivative $\hat{R}'$ is also in $L^2(\mathbb{R}_+, r dr)$; $H_R$ is a real inner product space when equipped with the inner product which induces the norm $\| \cdot \|_R$ defined by

$$\| \bar{R} \|_R^2 = \int_0^\infty r dr \left\{ [\hat{R}'(r)]^2 + \hat{R}^2(r) \right\},$$

and we will verify presently that $H_R$ is in fact a Hilbert space (i.e. it is complete).

On the other hand we will take $\bar{S}$ to belong to the space $H_S$ which consists of those real-valued continuous functions $\bar{S}$ on $\mathbb{R}_+$ for which the function $r^{-1}\bar{S}' \rightarrow r^{-1} \bar{S}(r)$ is bounded and for which the distributional derivative $\hat{S}'$ belongs to $L^2(\mathbb{R}_+, r^{-1} dr)$; $H_S$ is a real inner product space when equipped with the inner product which induces the norm $\| \cdot \|_S$. 
defined by

\[ \| \mathbf{\tilde{S}} \|_S \leq \int_0^\infty r^{-1} dr \left[ \mathbf{\tilde{S}}(r) \right]^2 \]

(Clearly \( \mathbf{\tilde{S}} = 0 \) if \( \| \mathbf{\tilde{S}} \|_S = 0 \) because \( \mathbf{\tilde{S}} \) is continuous and \( \mathbf{\tilde{S}}(r) \to 0 \) as \( r \to 0 \)). We show below that \( H \) is a Hilbert space. Finally, define \( H \) to consist of pairs \((R, S)\) with \( R = \mathbf{\tilde{R}} + \mathbf{\tilde{R}} \) for some \( \mathbf{\tilde{R}} \) in \( H_R \) and with \( S = \mathbf{\tilde{S}} + \mathbf{\tilde{S}} \) for some \( \mathbf{\tilde{S}} \) in \( H_S \). (Thus \( H \) is not a Hilbert space, but rather an affine Hilbert space; this will cause no difficulty.)

Clearly \( F \), as given by (II.3-4), is well defined as a Lebesque integral for any \((R, S)\) in \( H \), although it may assume the value \( +\infty \) at some points in \( H \). (Here we need to suppose that \( V \) is measurable and nonnegative, so that the integrand for \( F \) is measurable and nonnegative.)

Since \( F(R, S) \) is finite, the infimum \( \inf F \) of \( F \) over \( H \) is finite.

It is less clear, however, that an element \((R_{\min}, S_{\min})\) of \( H \) which minimizes \( F \) should be expected to obey the correct boundary conditions; the heuristic reasoning is as follows. The finiteness of \( \| S_{\min} - \mathbf{\tilde{S}} \|_S \) guarantees that \( S_{\min}(r) \to 0 \) as \( r \to 0 \), as shown in Lemma II.3.1 below. This fact together with the fact that the third term

\[ \int_0^\infty r dr \cdot \frac{1}{2} r^{-2} R_{\min}^2(r) \left( S_{\min}^2(r) + \gamma \right)^2 \]

in \( F(R_{\min}, S_{\min}) \) is finite indicates that \( R_{\min}(r) \) should vanish as \( r \to 0 \). Similarly one expects that \( R_{\min}(r) \to 1 \) as \( r \to \infty \) because \( \| R_{\min} - \mathbf{\tilde{R}} \|_R \) is finite, and that \( S_{\min}(r) \to -\gamma \) as \( r \to \infty \), again because the third term in \( F(R_{\min}, S_{\min}) \) is finite. These arguments will be made precise in II.4 and II.5. (See also the comments after Theorem III.2.)

Let us now demonstrate some properties of the spaces \( H_R \) and \( H_S \).
Lemma II.3.1. If \( S \in H_S \), then

\[
\sup_{r \in R_+} |r^{-1} \tilde{S}(r)| \leq 2^{-\frac{1}{2}} \| \tilde{S} \|_S.
\]

Proof: Let \( r \in R_+ \). By the Schwarz inequality

\[
\int_0^r \int_{r'} r^{\frac{1}{2}} \left( \int_0^r |\tilde{S}'(r')|^2 \right)^{\frac{1}{2}} \mathrm{d}r' \leq \left( \int_0^r \left( \int_0^r (r')^{-1} \int_{r'}^r |\tilde{S}'(r')|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}.
\]

so \( \tilde{S}' \) is integrable over \( ]0, r[ \). As a result

\[
\tilde{S}(r) = \int_0^r \int_{r'} r^{\frac{1}{2}} \tilde{S}'(r')
\]

because the distributional derivatives of each side coincide and both sides vanish as \( r \to 0 \). (Recall that the distributional derivative of an absolutely continuous function is given by its classical derivative, and that the classical derivative of the antiderivative of a locally integrable function equals the integrand almost everywhere [KF].) Therefore,

\[
|r^{-1} \tilde{S}(r)| \leq r^{-1} \int_0^r \int_{r'} |\tilde{S}'(r')| \leq 2^{-\frac{1}{2}} \| \tilde{S} \|_S.
\]

Lemma II.3.2. \( H_R \) and \( H_S \) are Hilbert spaces.

Proof: We present only the proof for \( H_S \); the proof for \( H_R \) is analogous (and more closely related to the corresponding proof for Sobolev spaces [A]).

Suppose \( j \to \tilde{S}_j \) is a Cauchy sequence in \( H_S \). By definition, \( \tilde{S}_j' \) is a Cauchy sequence in \( L^2(R_+, r^{-1} \mathrm{d}r) \); since \( L^2(R_+, r^{-1} \mathrm{d}r) \) is complete there exists an \( \tilde{S}(1) \) in \( L^2(R_+, r^{-1} \mathrm{d}r) \) to which \( \tilde{S}_j' \) converges. By the previous lemma, \( j \to r^{-1} \tilde{S}_j \) is a Cauchy sequence in \( C_b(R_+) \); since \( C_b(R_+) \) is complete, \( r^{-1} \tilde{S}_j \to r^{-1} \tilde{S} \) uniformly for some continuous \( \tilde{S} \).
such that $r^{-1} \mathcal{S}$ is bounded. We conclude the proof by showing that 

$\mathcal{S}^\prime = \mathcal{S}^{(1)}$ as distributions over $\mathbb{R}_+$. Indeed, given any $\varphi \in C_0^\infty(\mathbb{R}_+)$,

$$
\int_0^\infty \left( \mathcal{S}^{(1)}(r) \varphi(r) \right) dr = \lim_{j \to \infty} \int_0^\infty \mathcal{S}_j(r) \varphi(r) dr
= \lim_{j \to \infty} - \int_0^\infty \mathcal{S}_j^\prime(r) \varphi(r) dr
= - \int_0^\infty \mathcal{S}(r) \varphi(r) dr.
$$

Thus $\mathcal{S}^\prime = \mathcal{S}^{(1)} \in L^2(\mathbb{R}_+, r^{-1} dr)$ and $\mathcal{S}_j \to \mathcal{S}^\prime$ in $L^2(\mathbb{R}_+, r^{-1} dr)$, i.e. $\mathcal{S} \in H_S$ and $\mathcal{S}_j \to \mathcal{S}$ in $H_S$. \qed

The next result allows us to modify a minimizing sequence for $F$ so as to obtain one which has special properties crucial to extracting a convergent subsequence.

**Lemma II.3.3.** Suppose $(R, S) \in \mathcal{H}$. Define $R^{\text{mod}}$ to be the function

$$
R \mapsto \min \left\{ |R(r)|, 1 \right\}
$$

and $S^{\text{mod}}$ to be the function

$$
R \mapsto - \min \left\{ \int_0^R dr' |S'(r)|, n \right\}.
$$

Then the modified pair $(R^{\text{mod}}, S^{\text{mod}})$ is a member of $\mathcal{H}$ and

$$
F(R^{\text{mod}}, S^{\text{mod}}) \leq F(R, S).
$$

Moreover, $0 \leq R^{\text{mod}} \leq 1$, $-n \leq S^{\text{mod}} \leq 0$, and $S^{\text{mod}}$ is a non-increasing function.
proof: First we derive a formula for the derivative of the absolute value of a function. Suppose the real-valued function $T$ on $\mathbb{R}_+$ and its distributional derivative $T'$ are locally integrable. Then $T$ is absolutely continuous (by the fundamental theorem of calculus) and hence so is $|T|_{\varepsilon} = [T^2 + \varepsilon^2]^{1/2}$. Taking derivatives gives, by the Leibnitz rule, that $|T|'_{\varepsilon} = T \cdot |T|_{\varepsilon}^{-1} T'$. Define the function $\sgn$ by $\sgn(x) = 1, 0, -1$ according as $x > 0, = 0, < 0$; then we find that the formula $|T|' = \sgn T \cdot T'$ holds in the sense of distributions: by the dominated convergence theorem,

$$- \int_0^\infty \, dr \, |T(r)| \, \psi'(r) = \lim_{\varepsilon \to 0} - \int_0^\infty \, dr \, |T(r)|_{\varepsilon} \, \psi'(r)$$

$$= \lim_{\varepsilon \to 0} \int_0^\infty \, dr \, T(r) \cdot |T(r)|_{\varepsilon}^{-1} T'(r) \, \psi(r)$$

$$= \int_0^\infty \, dr \, \sgn (T(r)) \cdot T'(r) \, \psi(r)$$

for any $\psi \in C_c^\infty (\mathbb{R}_+)$. (Cf. the proof of Kato's inequality [RS].)

Let us estimate $R^\text{mod} - \hat{R}$: since $\min \{a, b\} = \frac{1}{2}[a+b - |a-b|],

$$|R^\text{mod} - \hat{R}| = \left| \frac{1}{2} \left[ |R| + 1 - |R|-1 \right] - \hat{R} \right|$$

$$\leq \frac{1}{2} \left| |R| - \hat{R} \right| + \frac{1}{2} \left| |R|-1 \right| - (1-\hat{R})$$

$$\leq \left| |R| - \hat{R} \right| \leq |R - \hat{R}|$$

by the triangle inequality, so $R^\text{mod} - \hat{R} \in L^2(\mathbb{R}_+, rdr)$. Furthermore, $(R^\text{mod})' = \frac{1}{2}[|R| + 1 - |R|-1]' = \frac{1}{2}(1 - \sgn(|R| - 1)) \cdot \sgn R \cdot R'$, so $|(R^\text{mod})'| \leq |R'|$. But $\hat{R}' \in C_c^\infty (\mathbb{R}_+)$, so $R'$ and therefore $(R^\text{mod} - \hat{R})'$ belong to $L^2(\mathbb{R}_+, rdr)$; thus $R^\text{mod} - \hat{R} \in H_R$. On the
other hand,

\[ r^{-1} \int_0^r dr' |S'(r')| \leq 2^{-1/2} \left[ \int_0^\infty (r')^{-1} dr' \left[ S'(r') \right]^2 \right]^{1/2} \]

\[ \leq 2^{-1/2} \left\{ \left[ \int_0^\infty (r')^{-1} dr' \left[ S'(r') \right]^2 \right]^{1/2} + \| S - \tilde{S} \|_S \right\} \]

by the Schwarz and triangle inequalities, so that the function \( r \mapsto \int_0^r dr' |S'(r')| \) belongs to \( H_S \), its derivative being \( |S'| \). Therefore, by an argument similar to the one above, \( S^{\text{mod}} - \tilde{S} \in H_S \). Also, \( (S^{\text{mod}})'(r) = - \left[ 1 - \text{sgn} \left( \int_0^r dr' |S'(r')| - n \right) \right] |S'(r)| \), so that \( 0 \leq (S^{\text{mod}})'(r) \leq |S'| \). Finally, \( V(R^{\text{mod}}) \leq V(R) \) because \( V(|R|) = V(R) \) and \( V(1) = 0 \leq V(R) \); and \( 0 \leq S^{\text{mod}} + n \leq |S+n| \) because \( S^{\text{mod}} + n = \max \{ n - \int_0^r dr' |S'(r')|, 0 \} \) and \( n - \int_0^r dr' |S'(r')| \leq S(r) + n \).

Thus \( F(R^{\text{mod}}, S^{\text{mod}}) \leq F(R,S) \), and \( R^{\text{mod}} \) and \( S^{\text{mod}} \) have the desired properties. \( \square \)

Lemma II.3.4 is a simple instance of a Sobolev embedding theorem [A]; it will be used to prove that the weak limit of a minimizing sequence for \( F \) actually minimizes \( F \).

**Lemma II.3.4.** Let \( a,b \in \mathbb{R} \) with \( a < b \) and define \( H_{a,b} \) to be the space of those \( T \in L^2([a,b[, dr) \) whose distributional derivative \( T' \) belongs to \( L^2([a,b[, dr) \); equip \( H_{a,b} \) with the norm \( \| \cdot \|_{a,b} \) given by

\[ \| T \|_{a,b}^2 = \int_a^b dr \left\{ \left[ T'(r) \right]^2 + T^2(r) \right\} . \]

If \( T \in H_{a,b} \) then \( T \) is (identifiable with) a continuous function on \([a,b[ \) and

\[ \sup_{x \in [a,b[} |T(x)| + \sup_{x,y \in [a,b[, x \neq y} \frac{|T(x) - T(y)|}{|x - y|^{1/2}} \leq \text{const.} \| T \|_{a,b} . \]

**proof:** Suppose that \( x,y \in \mathbb{R} \) with \( x < y \); then
\[ T(y) = T(x) + \int_x^y dz \ T'(z) \]

because the Schwarz inequality shows that \( T' \) is locally integrable.

Therefore

\[ |T(y)| \leq |T(x)| + \int_x^y dz \ |T'(z)| \]

by integrating both sides with respect to \( x \) and using the Schwarz inequality

it follows

\[ (y-x) |T(y)| \leq (y-x)^{1/2} \left[ \int_x^y dx \ T^2(x) \right]^{1/2} + (y-x)^{3/2} \left[ \int_x^y dz \ [T'(z)]^2 \right]^{1/2}. \]

On the other hand,

\[ |T(y) - T(x)| \leq \int_x^y dz \ |T'(z)| \leq (y-x)^{1/2} \left[ \int_x^y dz \ [T'(z)]^2 \right]^{1/2}. \]

Combining these last two inequalities concludes the proof. \[ \]

Finally we can proceed with the proof of the main result of this section.

**Theorem II.3.5.** There exists a pair \((R_{\min}, S_{\min})\) in \( H \) such that

\[ F(R_{\min}, S_{\min}) = \inf_{(R,S) \in H} F(R,S) \]

and, moreover, such that \( 0 \leq R_{\min} \leq 1, -n \leq S_{\min} \leq 0 \), and such that \( S_{\min} \)

is a non-increasing function.

**proof:** Pick a sequence \( j \to (R_j, S_j) \) in \( H \) such that

\[ \lim_{j \to \infty} F(R_j, S_j) = \inf_{(R,S) \in H} F(R,S); \]

we may assume that \( F(R_j, S_j) \) is finite for all \( j \). In addition we may

assume, without loss of generality, that \( 0 \leq R_j \leq 1, -n \leq S_j \leq 0 \), and

\( S_j \) is a non-increasing function for each \( j \). This is because we may modify

\((R_j, S_j)\) as in Lemma II.3.3 without increasing the value of \( F(R_j, S_j) \).
(This idea of modifying a minimizing sequence has been used, e.g., in [CGM],
[GMGT], [L], and [St]. Here the modification of the \( S_j \) is only necessary to
show that the minimizing \( S_{\min} \) has certain properties, properties which we
will also obtain directly from the differential equation that we prove \( S_{\min} \)
to satisfy. In contrast, the modification of \( R_{\min} \) is crucial to the following
argument.) Note that \( R \rightarrow (R - 1)^2 V(R) \) is a strictly positive, continuous
function on the compact set \([0,1]\), since \( V(R) = \frac{1}{2} V''(1) \cdot (R-1)^2 + o(R-1)^2 \)
as \( R \rightarrow 1 \) and \( V''(1) > 0 \), so that there is a constant \( \alpha > 0 \) such that

\[
V(R) \geq \alpha (R-1)^2
\]

if \( 0 \leq R \leq 1 \). Thus we obtain the inequality

\[
F(R_j, S_j) \geq \frac{1}{2} \left( \left\{ \int_0^\infty r dr \left[ \tilde{R}_j'(r) \right]^2 \right\}^{1/2} - \left\{ \int_0^\infty r dr \left[ \dot{R}_j'(r) \right]^2 \right\}^{1/2} \right)^2
\]

\[
+ \frac{1}{2} \left( \left\{ \int_0^\infty r^{-1} dr \left[ \tilde{S}_j'(r) \right]^2 \right\}^{1/2} - \left\{ \int_0^\infty r^{-1} dr \left[ \dot{S}_j'(r) \right]^2 \right\}^{1/2} \right)^2
\]

\[
+ \alpha \left( \left\{ \int_0^\infty r dr \left[ \tilde{R}_j'(r) \right]^2 \right\}^{1/2} - \left\{ \int_0^\infty r dr \left[ \dot{R}_j'(r) \right]^2 \right\}^{1/2} \right)^2
\]

by using the triangle inequality, where we have written \( R_j = \tilde{R}_j + \tilde{R}_j \) and
\( S_j = \tilde{S}_j + \tilde{S}_j \). This implies that the \( \tilde{R}_j \) and \( \tilde{S}_j \) lie inside closed balls
in \( H_R \) and \( H_S \), respectively; but these closed balls are weakly compact
by the Banach-Alaoglu theorem [RS], since Hilbert spaces are reflexive.

By passing to subnets we may suppose that \( \tilde{R}_j \rightarrow \tilde{R}_\infty \) and \( \tilde{S}_j \rightarrow \tilde{S}_\infty \) weakly
in \( H_R \) and \( H_S \), respectively, for some \( \tilde{R}_\infty \in H_R \) and \( \tilde{S}_\infty \in H_S \). Set
\( R_{\min} = \tilde{R} + \tilde{R}_\infty \) and \( S_{\min} = \tilde{S} + \tilde{S}_\infty \).

Now by Lemma II.3.4 the nets \( j \rightarrow \tilde{R}_j \) \( 2^{-1} \), \( j \rightarrow \tilde{S}_j \) \( 2^{-1} \), \( j \rightarrow \tilde{R}_j \) \( 2^{-1} \), \( j \rightarrow \tilde{S}_j \) \( 2^{-1} \), \( j \rightarrow \tilde{R}_j \) \( 2^{-1} \), \( j \rightarrow \tilde{S}_j \) \( 2^{-1} \)
are uniformly bounded and equicontinuous; thus, by the Arzela-Ascoli theorem
[RS], there are subsequences \( j \rightarrow \tilde{R}_j^{(2)} \) and \( j \rightarrow \tilde{S}_j^{(2)} \) of these nets which
converge uniformly over $\{2^{-1}, 2\}$. Continuing by induction we extract families of successive subsequences $j \rightarrow \tilde{R}_j^{(k)}$ and $j \rightarrow \tilde{S}_j^{(k)}$, indexed by integers $k \geq 2$, such that $\tilde{R}_j^{(k)} \rightarrow \tilde{R}_\infty$ and $\tilde{S}_j^{(k)} \rightarrow \tilde{S}_\infty$ uniformly over $\{k^{-1}, k\}$. Therefore, by passing to the diagonal subsequences of these families we may assume that $\tilde{R}_j \rightarrow \tilde{R}_\infty$ and $\tilde{S}_j \rightarrow \tilde{S}_\infty$ uniformly over any compact subset of $\tilde{R}_\infty$. In particular, $R_{\min}$ and $S_{\min}$ have the properties assumed of each of the $R_j$ and $S_j$.

To conclude the proof note that: the maps

$$\tilde{R} \mapsto \int_0^\infty r \, dr \cdot \frac{1}{2} \left[ \tilde{R}'(r) + \dot{\tilde{R}}'(r) \right]^2$$

and

$$\tilde{S} \mapsto \int_0^\infty r \, dr \cdot \frac{1}{2} r^{-2} \left[ \tilde{S}'(r) + \dot{\tilde{S}}'(r) \right]^2$$

are strongly continuous and convex, hence are weakly lower semicontinuous [V], so that

$$\int_0^\infty r \, dr \cdot \frac{1}{2} \left[ R_{\min}'(r) \right]^2 \leq \lim_{j \to \infty} \int_0^\infty r \, dr \cdot \frac{1}{2} \left[ R_j'(r) \right]^2$$

and

$$\int_0^\infty r \, dr \cdot \frac{1}{2} r^{-2} \left[ S_{\min}'(r) \right]^2 \leq \lim_{j \to \infty} \int_0^\infty r \, dr \cdot \frac{1}{2} r^{-2} \left[ S_j'(r) \right]^2$$

and by Fatou's lemma

$$\int_0^\infty r \, dr \left\{ \frac{1}{2} r^{-2} R_{\min}^2(r) \left( S_{\min}(r) + n \right)^2 + V(R_{\min}(r)) \right\} \leq \liminf_{j \to \infty} \int_0^\infty r \, dr \left\{ \frac{1}{2} r^{-2} R_j^2(r) \left( S_j(r) + n \right)^2 + V(R_j(r)) \right\}$$
Thus
\[ F(R_{\text{min}}, S_{\text{min}}) \leq \inf_{(R, S) \in \mathcal{H}} F(R, S). \square \]

We have thus found a minimum \((R_{\text{min}}, S_{\text{min}})\) of \(F\) over \(\mathcal{H}\). In our introductory remarks we argued that such a minimum should in some sense solve our boundary-value problem (II.3-1-II.3-3); the sense in which this is true is made precise in the following

**Definition II.3.6.** An element \((R, S)\) of \(\mathcal{H}\) with \(F(R, S)\) finite is said to be a weak solution of (II.3-1) if and only if

\[
\int_{\infty}^0 \! r \, dr \, R'(r) \, \tilde{R}'(r) + \int_0^\infty \! r \, dr \, \left[ r^{-2} \left( S(r) + n \right)^2 R(r) + V'(R(r)) \right] \tilde{R}(r) = 0
\]

and

\[
\int_{\infty}^0 \! r \, dr \, r^{-2} S'(r) \, \tilde{S}'(r) + \int_0^\infty \! r \, dr \, r^{-2} \tilde{R}^2(r) \left( S(r) + n \right) \tilde{S}(r) = 0
\]

for all \(R \in \mathcal{H}_R\) and \(S \in \mathcal{H}_S\) such that \(\tilde{R}(r)\) and \(\tilde{S}(r)\) vanish for all sufficiently large \(r\) and such that \(R(r) = O(r)\) as \(r \to 0\).

**Theorem II.3.7.** A minimum \((R, S)\) of \(F\) over \(\mathcal{H}\) is a weak solution of (II.3-1).

**proof:** First we prove that \(|R| \leq 1\); (cf. a similar argument in [GJ]). Suppose, to the contrary, that \(R(r_0) > 1\) for some \(r_0 \in \mathbb{R}_+\). Consider the function \(R^{\text{mod}}\) defined by

\[
R^{\text{mod}}(r) = \min \{ R(r), 1 \}.
\]

Then by the proof of Lemma II.3.3, \(|(R^{\text{mod}})'| \leq |R'|\), so that \((R^{\text{mod}}, S) \in \mathcal{H}\)
and

\[ F(R^{\text{mod}}, S) < F(R, S), \]

the inequality being strict because \(1 = R^{\text{mod}}(r) < R(r)\) for all \(r\) in an interval about \(r_0\) and \(V(1) < V(R_1)\) if \(R_1\) is slightly larger than 1. This contradiction shows that \(R \leq 1\), and similarly we can show that \(-1 \leq R\).

Let \(t \in \mathbb{R}\) and let \(\tilde{R}\) and \(\tilde{S}\) be as in Definition II.3.6. Note that \(\tilde{R}\) is uniformly bounded, say \(|\tilde{R}| \leq \alpha\). Then: \((R + t \tilde{R}, S) \in \mathbb{H}\
\]

\(F(R + t \tilde{R}, S)\) is finite since \(\tilde{R}(r) = O(r)\) as \(r \to 0\), \(R + t \tilde{R}\) is uniformly bounded, and \(\tilde{R}(r)\) vanishes for all large \(r\); and \(F(R + t \tilde{R}, S) \geq F(R, S)\)

because \((R, S)\) minimizes \(F\) over \(\mathbb{H}\). If we write \(\tilde{V}(t, r) = V(R(r) + t\tilde{R}(r))\)

\(- [V(R(r)) + t V'(R(r)) \tilde{R}(r)]\) then

\[
F(R + t \tilde{R}, S) = F(R, S) \\
+ t \int_0^\infty r \, dr \ R'(r) \tilde{R}'(r) + t \int_0^\infty r \, dr \ [r^{-2}(S(r) + \eta)^2 \tilde{R}(r) + V'(R(r))\tilde{R}(r)] \\
+ t^2 \int_0^\infty r \, dr \ \left\{ \frac{1}{2} \tilde{R}'(r) \right\}^2 + \frac{1}{2} r^{-2} \tilde{R}'(r) (S(r) + \eta)^2 \right\} + \int_0^\infty r \, dr \ \tilde{V}(t, r) \\
\]

(noting that \(r \mapsto r \tilde{V}'(R(r)) \tilde{R}(r)\) is integrable). Because \(V\) is twice continuously differentiable we conclude from Taylor's theorem that

\[
\left| \tilde{V}(t, r) \right| \leq \frac{1}{2} t^2 \tilde{R}'(r) \left. \sup_{-\alpha \leq |R_1| \leq \alpha \vert t\vert} \right| V''(R_1) \right|.
\]

Thus

\[
\lim_{t \to 0} \frac{1}{t} \left[ F(R + t \tilde{R}, S) - F(R, S) \right] = \int_0^\infty r \, dr \ \tilde{R}'(r) \tilde{R}'(r) \\
+ \int_0^\infty r \, dr \ \left[ r^{-2}(S(r) + \eta)^2 \tilde{R}(r) + V'(R(r)) \right] \tilde{R}(r); \\
\]

24.
but this limit must vanish since \( \frac{1}{t}[F(R + t\vec{R}, S) - F(R, S)] \) is non-negative if \( t > 0 \) and is non-positive if \( t < 0 \).

On the other hand: \( (R, S + t\vec{S}) \in \mathcal{H} \); \( F(R, S + t\vec{S}) \) is finite because \( \vec{S}(r) \) vanishes for large \( r \); and \( F(R, S + t\vec{S}) \geq F(R, S) \). Also,

\[
F(R, S + t\vec{S}) = F(R, S) + t \int_0^\infty r \, dr \cdot r^{-2} \vec{S}'(r) \vec{S}'(r) + t \int_0^\infty r \, dr \cdot r^{-2} \vec{S}'(r) (S(r) + v) \vec{S}(r) + t^2 \int_0^\infty r \, dr \cdot \frac{1}{2} r^{-2} \vec{S}'(r) \vec{S}'(r). 
\]

Just as before we conclude that

\[
0 = \int_0^\infty r \, dr \cdot r^{-2} S'(r) \vec{S}'(r) + \int_0^\infty r \, dr \cdot r^{-2} \vec{S}'(r) (S(r) + v) \vec{S}(r). \]

Of course it remains to be seen whether a weak solution of (II.3-1) actually solves the boundary-value problem we are studying; this question is answered in II.4 and II.5.
II.4 The Abelian Higgs Model - Regularity of Solutions

In this section we wish to prove that a weak solution of (II.3-1) (the existence of which was demonstrated in II.3) gives rise to Higgs and gauge fields $\phi$ and $A$ that satisfy the equations

$$
\begin{align*}
\partial_A \partial_A \phi + \nabla' \left( \frac{\phi}{\|\phi\|} \right) &= 0 \\
\text{and} \\
\partial^* \partial A + \frac{1}{2\pi} \left( \bar{\phi} \partial_A \phi - \phi \partial_A \bar{\phi} \right) &= 0.
\end{align*}
$$

(II.4-1)

Here $\phi$ and $A$ are defined (on $\mathbb{R}^2 \setminus \{0\}$) by

$$
\begin{align*}
\phi(x, y) &= R(r) \exp \left( \imath \eta \theta \right) \\
\text{and} \\
A(x, y) &= S(r) \imath \theta,
\end{align*}
$$

(II.4-2)

where $r$ and $\theta$ are the usual polar coordinates for $\mathbb{R}^2 \setminus \{0\}$. The proof is based on establishing two facts: first, that $\phi$ and $A$ as given by (II.4-2) are $C^2$ in the region $\mathbb{R}^2 \setminus \{0\}$ and satisfy (II.4-1) there; and second, that $\phi$ and $A$ extend to all of $\mathbb{R}^2$ in a $C^2$ manner. From these two facts it follows that (II.4-1) holds for the extended $\phi$ and $A$, for it holds on $\mathbb{R}^2 \setminus \{0\}$, so it holds at the origin by the continuity of the quantities on the left in (II.4-1).

It is clear from the analysis in II.1 that to establish the first fact we need only show that $R$ and $S$ are $C^2$ on $\mathbb{R}_+$ and satisfy

$$
-R''(r) - r^{-1} R'(r) + r^{-2} \left( S(r) + n \right)^2 R(r) + \nabla' \left( R(r) \right) = 0
$$

and
\[-S''(r) + r^{-1} S'(r) + R^2(r) (S(r) + \eta) = 0;\]

we will prove this by deriving integral formulae for \(R\) and \(S\). To establish the second fact we will reduce the problem of extending \(\phi\) and \(A\) to proving that \(R(r)\) and \(S(r)\) vanish sufficiently rapidly as \(r \to 0\), a condition that the integral formulae will show to be satisfied. Thus the integral formulae are the heart of the proof.

Let us introduce the truncated Green functions with which we will obtain the integral formulae for \(R\) and \(S\). Let \(0 < r_0 < r_1\) and define \(G^r_{R_1}\) and \(G^r_{S_1}\) on \(\mathbb{R}_+ \times \mathbb{R}_+\) by

\[
G^r_{R_1}(r_0, r) = \begin{cases} 
\frac{1}{2n} \left(\frac{r}{r_0}\right)^n & \text{if } r \leq r_0 \\
\frac{1}{2n} \left(\frac{r_0}{r}\right)^n \chi_{r_1}(r) & \text{if } r > r_0
\end{cases}
\]

and

\[
G^r_{S_1}(r_0, r) = \begin{cases} 
\frac{1}{2} r^2 & \text{if } r \leq r_0 \\
\frac{1}{2} r_0^2 \chi_{r_1}(r) & \text{if } r > r_0
\end{cases}
\]

where \(\chi_{r_1}\) is a \(C^\infty\) function on \(\mathbb{R}_+\) such that \(\chi_{r_1}(r) = 1\) for \(r \leq r_1\) and \(\chi_{r_1}(r) = 0\) for \(r > r_1 + 1\). It is easy to check that \(G^r_{R_1}(r_0, \cdot) \in H_R\) and \(G^r_{S_1}(r_0, \cdot) \in H_S\) (because they have been truncated at infinity). Moreover we have the following identities.

**Lemma II.4.1.** If \((R, S) \in H\) is such that \(F(R, S)\) is finite, then there are constants \(\alpha\) and \(\beta\) such that
\[ R(r_o) - \int_0^\infty r \, dr \left\{ R'(r) \frac{\partial G_r^{I_r}(r_o, r)}{\partial r} + r^{-2} h^2 R(r) \, G_r^{I_r}(r_o, r) \right\} + \alpha r_o^n = 0 \]

and

\[ S(r_o) - \int_0^\infty r \, dr \cdot r^{-2} S'(r) \frac{\partial G_r^{I_r}(r_o, r)}{\partial r} + \beta r_o^2 = 0 \]

for all positive \( r_o < r_1 \).

**proof:** First let us prove that \( R(r) \to 0 \) as \( r \to 0 \). In fact, \( R \) is absolutely continuous since \( R' \) is locally integrable, so that

\[ [R^2]' = 2R \cdot R' \text{ as distributions.} \]

Also by the Schwarz inequality,

\[ \int_0^{r_o} dr \left| R(r) \right| \left| R'(r) \right| \leq \left[ \int_0^{r_o} dr \left[ R'(r) \right]^2 \right]^{1/2} \left[ \int_0^{r_o} r \, dr \cdot r^{-2} R^2(r) \right]^{1/2} ; \]

the second factor on the right side of this inequality is finite for small enough \( r_o \), because \( \int_0^\infty r \, dr \cdot r^{-2} R^2(r)(S(r) + n)^{1/2} \) is finite and because \( S(r) \to 0 \) as \( r \to 0 \). Therefore

\[ \lim_{r_o \to 0} R^2(r_o) = R^2(r_o) - \lim_{r_o \to 0} \int_0^{r_o} dr \cdot \frac{d}{dr} [R^2(r)] \]

exists; but this limit must be zero since \( \int_0^{r_1} r^{-1} dr R^2(r) \) is finite. Consequently,

\[ r_o^n R(r_o) = \int_0^{r_o} dr \cdot \frac{d}{dr} [R(r) \, r^n] \]

and

\[ r_o^{-n} R(r_o) = - \int_{r_o}^{\infty} dr \cdot \frac{1}{r} \left[ R(r) \, r^{-n} \chi_n^r(r) \right] . \]
Now we can calculate that

\[
\int_0^\infty r dr R'(r) \frac{\partial G'}{\partial r} (r, r) = \frac{1}{2} \int_0^{r_o} dr R'(r) \left( \frac{r}{r_o} \right)^n - \frac{1}{2} \int_{r_o}^\infty dr R'(r) \left( \frac{r}{r_o} \right)^n \chi'_{r_o}(r) + \int_{r_o}^\infty r dr R'(r) \frac{1}{2n} \left( \frac{r}{r_o} \right)^n \chi'_{r_o}(r)
\]

\[
= \frac{1}{2} \int_0^{r_o} dr \left[ \frac{1}{dr} \left( R(r) \left( \frac{r}{r_o} \right)^n \right) \right] - \int_0^{r_o} r dr \cdot r^{-2} n^2 R(r) \frac{1}{2n} \left( \frac{r}{r_o} \right)^n \chi'_{r_o}(r)
\]

\[
- \frac{1}{2} \int_{r_o}^\infty dr \cdot \frac{1}{dr} \left[ R(r) \left( \frac{r}{r_o} \right)^n \chi'_{r_o}(r) \right] - \int_{r_o}^\infty r dr \cdot r^{-2} n^2 R(r) \frac{1}{2n} \left( \frac{r}{r_o} \right)^n \chi'_{r_o}(r)
\]

\[
+ r_o^n \int_{r_o}^\infty dr R(r) \frac{1}{2} r^{-2} n \chi'_{r_o}(r) + r_o^n \int_{r_o}^\infty r dr R'(r) \frac{1}{2n} r^{-n} \chi'_{r_o}(r)
\]

\[
= R(r_o) - \int_0^\infty dr \cdot r^{-2} n^2 R(r) G'_{\mathcal{R}}(r_o, r) + \alpha r_o^n
\]

where \( \alpha \) is a constant independent of \( r_o \). In the same manner,

\[
\int_0^\infty r dr \cdot r^{-2} S'(r) \frac{\partial G'}{\partial r} (r, r) = \int_0^{r_o} dr S'(r) + \int_{r_o}^\infty r dr \cdot r^{-2} S'(r) \frac{1}{2} r_o^2 \chi'_{r_o}(r)
\]

\[
= S(r_o) + \beta r_o^2
\]

where \( \beta \) is independent of \( r_o \). □

As an immediate corollary of Lemma II.4.1 and Definition II.3.6 we obtain the following integral formulae for weak solutions of (II.3-1).

**Lemma II.4.2.** Suppose that \((R, S)\) is a weak solution of (II.3-1). Then there are constants \( \alpha \) and \( \beta \) such that
\[ 0 = R(r_o) + \int_0^\infty r \, d\gamma \left\{ r^{-2} S'(r) \left( S(r) + 2n \right) R(e^{-\gamma}) + \sqrt{\gamma} \left( R(e^{-\gamma}) \right) \right\} G_e^r(r_o, r) + \alpha r_o^n \]

and

\[ 0 = S(r_o) + \int_0^\infty r \, d\gamma \cdot r^{-2} R^2(r) \left( S(r) + n \right) G_e^s(r_o, r) + \beta r_o^2 \]

for all positive \( r_o < r_1. \)

Our final lemma will be used to extend \( \phi \) and \( A, \) as defined by (II.4-2), to all of \( \mathbb{R}^2. \)

**Lemma II.4.3.** Let \( f \) be a \( C^2 \) function on \( \mathbb{R}_+ \) and define the \( C^2 \) function \( \psi \) on \( \mathbb{R}^N \) \( \setminus \{0\} \) by

\[ \psi(x) = x^i f(|x|) \]

where \( x^i \) is a Cartesian component of \( x. \) Then \( \psi \) can be extended to a \( C^2 \) function on \( \mathbb{R}^N \) if:

\[ \lim_{r \to 0^+} f(r) \text{ exists}; \]

\[ \lim_{r \to 0^+} f'(r) = 0; \]

and

\[ \lim_{r \to 0^+} r f''(r) = 0. \]

**proof:** First we show that if \( \psi \) and its first and second derivatives extend to continuous functions on \( \mathbb{R}^N, \) then the extension \( \hat{\psi} \) of \( \psi \) to \( \mathbb{R}^N \) is \( C^2. \) Clearly \( \hat{\psi} \) is \( C^2 \) on \( \mathbb{R}^N \setminus \{0\}, \) so we need to show that its first and second partial derivatives exist and are continuous at the origin.
Suppose \( \hat{\psi}_k \) extends \( \frac{\partial \psi}{\partial x} \) to \( \mathbb{R}^N \). Then the function \( s \mapsto \hat{\psi}(0, \ldots, (k), \ldots, 0) = \int_0^s dt \hat{\psi}_k(0, \ldots, t, \ldots, 0) \) on \( \mathbb{R} \) is continuous, and its derivative vanishes at any \( s \neq 0 \), so it must be constant. Thus \( \frac{\partial \hat{\psi}_k(0)}{\partial x} = \frac{d}{ds} \bigg|_{s=0} \hat{\psi}(0, \ldots, (k), \ldots, 0) \) exists and equals \( \hat{\psi}_k(0) = \lim_{x \to 0} \frac{\partial \hat{\psi}}{\partial x}(x) \), i.e. \( \frac{\partial \hat{\psi}_k}{\partial x} \) exists and is continuous at \( 0 \in \mathbb{R}^N \). Similarly the second partial derivatives of \( \hat{\psi} \) exist and are continuous at \( 0 \).

A simple calculation shows that for \( x \neq 0 \)

\[
\frac{\partial \psi}{\partial x^k}(x) = \delta^{jk} f'(|x|) + |x|^{-1} x^i x^k f''(|x|)
\]

and

\[
\frac{\partial^2 \psi}{\partial x^k \partial x^l}(x) = |x|^{-1} \left( \delta^{jk} x^l + \delta^{lj} x^k + \delta^{kl} x^j \right) f'(|x|) + |x|^{-3} x^i x^k x^l \left[ |x| f''(|x|) - f'(|x|) \right]
\]

Therefore our hypotheses guarantee that the derivatives of \( \psi \) have limits at the origin, so that they extend to continuous functions on \( \mathbb{R}^N \). Defining \( \psi(0) = 0 \) thus extends \( \psi \) to a \( C^2 \) function on \( \mathbb{R}^N \).

We now prove

**Theorem II.4.4.** Suppose \((R,S)\) is a weak solution of II.3-1). Then there exist \( C^2 \) fields \( \phi \) and \( A \) on \( \mathbb{R}^2 \) that extend the definition (II.4-2) and satisfy (II.4-1).

**proof:** Let us rewrite the integral formulae (II.4-3) of Lemma II.4.2 in the form
\[
R(r_0) + \int_0^{r_0} r \, dr \left\{ r^{-2} S(r) (S(r) + 2n) R(r) + V'(R(r)) \right\} \cdot \frac{1}{2n} \left( \frac{r}{r_0} \right)^n \\
+ \int_{r_0}^{r_1} r \, dr \left\{ r^{-2} S(r) (S(r) + 2n) R(r) + V'(R(r)) \right\} \cdot \frac{1}{2n} \left( \frac{r}{r} \right)^n \\
+ \alpha' r_0^n = 0
\]

and

\[
S(r_0) + \int_0^{r_0} r \, dr \cdot r^{-2} R(r) (S(r) + n) \cdot \frac{1}{2} r^2 \\
+ \int_{r_0}^{r_1} r \, dr \cdot r^{-2} R(r) (S(r) + n) \cdot \frac{1}{2} r_0^2 \\
+ \beta r_0^2 = 0
\]

where the constants \( \alpha' \) and \( \beta' \) are independent of \( r_0 \) and \( 0 < r_0 < r_1 \).

Since the integrands in these expressions are continuous on \( \mathbb{R}_+ \), we may calculate the derivatives of \( R \) and \( S \) to be

\[
R'(r_0) = n r_0^{-1} \int_0^{r_0} r \, dr \left\{ r^{-2} S(r) (S(r) + 2n) R(r) + V'(R(r)) \right\} \cdot \frac{1}{2n} \left( \frac{r}{r_0} \right)^n \\
- n r_0^{-1} \int_{r_0}^{r_1} r \, dr \left\{ r^{-2} S(r) (S(r) + 2n) R(r) + V'(R(r)) \right\} \cdot \frac{1}{2n} \left( \frac{r}{r} \right)^n \\
- n \alpha' r_0^{n-1}
\]

and

\[
S'(r_0) = -2 r_0^{-1} \int_{r_0}^{r_1} r \, dr \cdot r^{-2} R^2(r) (S(r) + n) \cdot \frac{1}{2} r_0^2 \\
- 2 \beta' r_0^2
\]

thus \( R' \) and \( S' \) are continuous on \( \mathbb{R}_+ \), and we may differentiate again to find that \( R \) and \( S \) satisfy (II.3-1) and are \( C^2 \) on \( \mathbb{R}_+ \). Therefore \( \phi \) and \( A \), as defined by (II.4-2), are \( C^2 \) on \( \mathbb{R}^2 \setminus \{0\} \) and satisfy (II.4-1) there.
It remains to show that $\phi$ and $A$ extend to $\mathbb{R}^2$ in a $C^2$ manner. First note that if $A_x$ and $A_y$ are the Cartesian components of $A$ then

$$A_x(x, y) = \frac{1}{y} \cdot \left( x^2 + y^2 \right)^{-\frac{1}{2}} \cdot S \left( (x^2 + y^2)^{\frac{1}{2}} \right)$$

and

$$A_y(x, y) = x \cdot \left( x^2 + y^2 \right)^{-\frac{1}{2}} \cdot S \left( (x^2 + y^2)^{\frac{1}{2}} \right);$$

to extend $A$ to $\mathbb{R}^2$ it therefore suffices to verify that the function $r \to r^{-2} S(r)$ satisfies the properties required of $f$ in Lemma II.4.3.

Similarly it suffices to demonstrate that the function $r \to r^{-n} R(r)$ satisfies these properties in order to extend $\phi$, since $n \geq 1$ and

$$\phi(x, y) = (x + iy)^{n-1} \cdot [x + iy] \cdot \left( x^2 + y^2 \right)^{-\frac{n}{2}} \cdot R \left( (x^2 + y^2)^{\frac{1}{2}} \right).$$

To this end we rewrite the integral formulae (II.4-3) as

$$r_0^{-k} R(r_0) + \int_0^{r_0} dr \left\{ r^{-2} S(r) \left( S(r) + 2n \right) + \frac{V'(R(r))}{R(r)} \right\} \cdot \frac{1}{2n} \left( \frac{r}{r_0} \right)^{n+k} \cdot r^{-k+1} R(r)$$

$$+ \int_{r_0}^{r_1} dr \left\{ r^{-2} S(r) \left( S(r) + 2n \right) + \frac{V'(R(r))}{R(r)} \right\} \cdot \frac{1}{2n} \left( \frac{r}{r_0} \right)^{n-k} \cdot r^{-k+1} R(r)$$

$$+ \alpha \left( \frac{r}{r_0} \right)^{n-k} = 0$$

and

$$r_0^{-2} S(r_0) + \int_0^{r_0} dr \cdot r^{-2} R^2(r) \left( S(r) + n \right) \cdot \frac{1}{2} \left( \frac{r}{r_0} \right)^2$$

$$+ \int_{r_0}^{r_1} dr \cdot r^{-2} R^2(r) \left( S(r) + n \right) \cdot \frac{1}{2} + \beta' = 0,$$

where $k$ is a non-negative integer. Just as in the proof of Lemma II.4.1,
\[ \int_{r_0}^{r_1} r \, dr \cdot r^{-2} R^2(r) |S(r) + n| \text{ is finite for small enough } r_1; \text{ therefore we may let } r_0 \to 0 \text{ in the second formula to show that } \lim_{r \to 0} r^{-2} S(r_0) \]
exists. The proof of Lemma II.4.1 also demonstrated that \( R(r) \to 0 \) as \( r \to 0 \). Therefore the function

\[ r \mapsto \left\{ r^{-2} S(r) \left( S(r) + 2n \right) + \frac{\sqrt[2]{R(r)}}{R(r)} \right\}, \]

which is continuous on \( \mathbb{R}^+ \) because \( V'(R_1) = V''(0)R_1 + o(R_1) \) as \( R_1 \to 0 \), has a limit as \( r \to 0 \). We conclude from the first formula with \( k = 1 \) that \( \lim_{r_0 \to 0} r_0^{-1} R(r_0) \) exists; continuing by induction on \( k \) we eventually find \( \lim_{r_0 \to 0} r_0^{-n} R(r_0) \) exists. Furthermore, differentiation of these formulae gives that

\[ \frac{d}{dr_0} \left[ r_0^{-n} R(r_0) \right] = \int_0^{r_0} dr \left\{ r^{-2} S(r) \left( S(r) + 2n \right) + \frac{\sqrt[2]{R(r)}}{R(r)} \right\} \left( \frac{r}{r_0} \right)^{2n+1} r^{-n} R(r) \]

\[ r_0 \frac{d^2}{dr_0^2} \left[ r_0^{-n} R(r_0) \right] = r_0 \left\{ r_0^{-2} S(r_0) \left( S(r_0) + 2n \right) + \frac{\sqrt[2]{R(r_0)}}{R(r_0)} \right\} r_0^{-n} R(r_0) \]

\[ - (2n + 1) \frac{d}{dr_0} \left[ r_0^{-n} R(r_0) \right], \]

\[ \frac{d}{dr_0} \left[ r_0^{-2} S(r_0) \right] = \int_0^{r_0} dr \, r^{-2} R^2(r) \left( S(r) + n \right) \left( \frac{r}{r_0} \right)^3, \]

and

\[ r_0 \frac{d^3}{dr_0^3} \left[ r_0^{-2} S(r_0) \right] = r_0^{-1} R^2(r_0) \left( S(r_0) + n \right) - 3 \frac{d}{dr_0} \left[ r_0^{-2} S(r_0) \right], \]

so the limits as \( r_0 \to 0 \) of these quantities vanish. Thus by Lemma II.4.3 the fields \( \phi \) and \( A \) have \( C^2 \) extensions to \( \mathbb{R}^2 \), also denoted by \( \phi \) and \( A \), and these extensions satisfy (II.4-1) on \( \mathbb{R}^2 \) by continuity.
To summarize, we have shown that weak solutions of (II.3-1) are $C^2$ solutions of the differential equation (II.3-1) and satisfy the boundary conditions (II.3-2) at the origin. In the next section we prove that weak solutions of (II.3-1) also satisfy the boundary conditions (II.3-3) at infinity.
II.5 The Abelian Higgs Model \- Behavior of Solutions

In the previous sections we have established the existence of finite-energy, isotropic solutions of the Abelian Higgs equation, though it remains to be seen whether these solutions have a well-defined vortex number.

Having written the gauge field $A$ in the form $A(x,y) = S(r) \, d\theta$, the vortex number is given by

$$
\frac{i}{2\pi} \int_{\mathbb{R}^2} (\nabla \times A) = - \lim_{r \to \infty} \frac{S(r)}{r} ;
$$

thus we wish to show that $S(r)$ has the limit $-n$ as $r \to \infty$, where $n$ is the integer appearing in the boundary-value problem of II.3. On the other hand we expect that the Higgs field should minimize its energy by seeking the minima of the Higgs potential; in fact, with $\phi(x,y) = R(r)e^{i\theta}$, we would like to verify that $R(r) \to 1$ as $r \to \infty$, since the first minimum of $V$ occurs at $R_0 = 1$. The result that the Higgs field does not vanish at infinity is the classical counterpart of the Higgs mechanism, whereby the corresponding quantum field acquires a non-zero vacuum expectation value; see the Conclusion for further discussion.

In the following we determine the asymptotic behavior of $R(r)$ and $S(r)$ as $r \to \infty$. We find not only that $R(r) \to 1$ and $S(r) \to -n$ as $r \to \infty$, but also that they tend to these limits exponentially fast. The characteristic lengths of these exponential approaches to limits define the classical counterparts of the inverse masses of the gauge and Higgs fields.

In our system of units the mass of the gauge field is 1, whereas the mass of the Higgs field is determined by the sharpness of the minimum in the Higgs potential at $R_0 = 1$. Indeed, if $[V''(1)]^2 < 2$, then $[V''(1)]^2$ is the mass of the Higgs field; but if $[V''(1)]^2 > 2$ the mass of the Higgs field
is 2. This fact, that the Higgs field's mass cannot exceed twice the gauge field's mass, corresponds to the existence, in the quantized version of this model, of a decay mode in which a Higgs particle decays into two gauge particles (photons); we will discuss this further in the Conclusion.

In order to determine the asymptotics of our fields we will find it convenient to work with the shifted fields $u$ and $v$ defined by

$$u(r) = r^{\lambda_2/2} (1 - R(r))$$

and

$$v(r) = r^{-\lambda_2/2} (v + S(r))$$

instead of with $R$ and $S$. Given that $R$ and $S$ satisfy (II.3-1), the differential equations satisfied by $u$ and $v$ are easily found to be

$$-u''(r) + \left[ \frac{V''(r)}{r^2} - \frac{V}{r^2} u(r) \right] u(r) = r^{-\lambda_2} v^2(r)$$

and

$$-v''(r) + \left[ 1 + \frac{3r^2}{4} - 2r^{-\lambda_2} u(r) + r^{-1} u^2(r) \right] v(r) = 0,$$

where we have written $V'(1 - R) = -V''(1 - R) + \hat{V}(R) R^2$. Note that $\hat{V}(R) = O(1)$ as $R \to 0$ by Taylor's theorem.

In the proof of Theorem II.5.3 we will show that $u(r)$ and $v(r)$ vanish as $r \to \infty$, assuming that $u$ and $v$ come from fields $R$ and $S$ for which $F(R,S)$ is finite. We will then be in a position to apply the following lemma, which shows that solutions of differential equations of
the form (II.5-2) decay exponentially at infinity, given that they vanish at infinity. The proof of the lemma is based on an elementary form of the maximum principle [PW].

**Lemma II.5.1.** Suppose $w$ is a $C^2$ solution on $\mathbb{R}_+$ of the differential equation

$$-w''(r) + \left(\kappa^2 + f(r)\right)w(r) = g(r),$$

where $f$ and $g$ are continuous functions on $\mathbb{R}_+$ and $\kappa > 0$. If $w$ and $f$ vanish at infinity, and

$$g(r) = O\left(\exp(-\lambda r)\right)$$

as $r \to \infty$ for some $\lambda > 0$, then for every positive $\varepsilon < 1$

$$w(r) = O\left(\exp\left[-\min\left\{\kappa(1-\varepsilon)^{1/2}, \lambda^{1/2} r\right\}\right]\right)$$

as $r \to \infty$.

**proof:** Fix $0 < \varepsilon < 1$. By hypothesis there is an $r_\varepsilon \in \mathbb{R}_+$ and a constant $\gamma_\varepsilon$ such that $\frac{1}{2}\varepsilon \kappa^2 + f(r) \geq 0$ for all $r \geq r_\varepsilon$ and such that $|g(r)| \leq \gamma_\varepsilon \exp(-\lambda r)$ for all $r \geq r_\varepsilon$. Define $\kappa_\varepsilon = \kappa(1-\varepsilon)^{1/2}$, $f_\varepsilon = \frac{1}{2}\varepsilon \kappa^2 + \gamma_\varepsilon^2$, and $\mu_\varepsilon = \min\{\kappa(1-\varepsilon)^{1/2}, \lambda\}$. Thus

$$-w''(r) + \left(\kappa_\varepsilon^2 + f_\varepsilon(r)\right)w(r) = g(r)$$

with $f_\varepsilon(r) \geq 0$ for all $r \geq r_\varepsilon$, and $g(r) \leq \gamma_\varepsilon \exp(-\mu_\varepsilon r)$ for all $r \geq r_\varepsilon$.

Define the function $\tilde{w}$ by
\[ \tilde{w}(r) = p \exp(-\kappa \tau r) + \kappa - \mu^2 \gamma \exp(-\mu \tau r) - \omega(r), \]

where \( p \geq 0 \) is chosen so that \( \tilde{w}(r) \geq 0 \). One easily verifies that

\[-\tilde{w}''(r) + \left( \kappa^2 + f(r) \right) \tilde{w}(r) = f(r) \left[ \tilde{w}(r) + \omega(r) \right] + \gamma \exp(-\mu \tau r) - q,\]

so that

\[-\tilde{w}''(r) + \left( \kappa^2 + f(r) \right) \tilde{w}(r) \geq 0 \]

for all \( r \geq r \). (because \( p \geq 0 \) and \( \kappa^2 - \mu^2 > 0 \)). We will show that

\( \tilde{w}(r) \geq 0 \) if \( r \geq r \). Suppose to the contrary that \( \tilde{w}(r_1) < 0 \) for some \( r_1 > r \); then because \( \tilde{w}(r) \geq 0 \) and \( \tilde{w}(r) \to 0 \) as \( r \to \infty \) there exists an \( r_{\min} > r \) at which \( \tilde{w} \) attains a strictly negative minimum. But then

\[ \tilde{w}''(r_{\min}) \leq \left( \kappa^2 + f(r_{\min}) \right) \tilde{w}(r_{\min}) < 0, \]

contradicting the fact that \( \tilde{w} \) attains a minimum at \( r_{\min} \); thus \( \tilde{w}(r) \geq 0 \) for all \( r \geq r \). This

inequality implies the bound

\[ \omega(r) \leq \text{const.} \exp(-\mu \tau r) \]

for all \( r \geq r \), since \( \kappa \geq \mu \). We may also apply this argument to \( -\omega \), whence we deduce the inequality

\[ -\omega(r) \leq \text{const.} \exp(-\mu \tau r) \]

for \( r \geq r \), so that \( \omega(r) = 0(\exp(-\mu \tau r)) \) as \( r \to \infty \). \( \square \)

As a result of applying this lemma to \( u \) and \( v \) we will see that

the equations satisfied by \( u \) and \( v \) resemble modified Bessel equations, differing only by perturbations which are exponentially small at infinity. Therefore we will compare \( u \) and \( v \) to the modified Bessel functions that approximately solve (II.5-2) to obtain more precise results about their asymptotic behavior. Let \( K_\nu \) denote the usual [BMP] modified Bessel
function of order \( \nu \) that is subdominant at infinity; \( K_\nu \) satisfies the modified Bessel equation

\[
- y''(x) + x^{-1} y'(x) + \left(1 + \nu^2 x^{-2}\right) y(x) = 0.
\]

We define \( k_\nu \) on \( \mathbb{R}_+ \) by

\[
k_\nu(r) = \left(2r/\pi^\nu\right)^{1/2} K_\nu(r);
\]

\( k_\nu \) satisfies the differential equation

\[
- w''(r) + \left(1 + \left[\nu^2 - \frac{1}{4}\right] r^{-2}\right) w(r) = 0.
\]

The following properties of \( k_\nu \) are used below: if \( \nu^2 \) is real, then \( k_\nu \) is a real-valued function such that

\[
k_\nu(r) = e^{-r} (1 + \mathcal{O}(r^{-1}))
\]

and

\[
k_\nu'(r) = -k_\nu(r) \cdot \left(1 + \mathcal{O}(r^{-2})\right)
\]

as \( r \to \infty \). (See Appendix II.)

Lemma II.5.2 is a typical application of the variation-of-parameters technique from the theory of ordinary differential equations. It is a slightly refined form of the WKB approximation that has been used in many contexts.

**Lemma II.5.2.** Consider the differential equation

\[
- w''(r) + \left(k^2 + \left[\nu^2 - \frac{1}{4}\right] r^{-2} + h(r)\right) w(r) = 0,
\]

where \( h \) is a continuous function on \( \mathbb{R}_+ \), \( \nu^2 \) is real, and \( k > 0 \).

If \( |h| \) is integrable at infinity then there exist \( C^2 \) solutions \( w^+ \) and \( w^- \) of this equation such that
\[ w^\pm = \exp(\pm \kappa r) \cdot (1 + o(1)) \]

and

\[ [w^\pm]'(r) = \pm \kappa w^\pm(r) \cdot (1 + o(1)) \]

as \( r \to \infty \). If in addition \( h(r) = o(\exp(-\mu r)) \) as \( r \to \infty \), where \( \mu > 0 \), then

\[ w^-(r) = \mathcal{K}_\nu(\pi r) \cdot (1 + o(\exp(-\mu r))) \]

and

\[ [w^-]'(r) = -\kappa w^-(r) \cdot (1 + O(r^{-2})) \]

**Proof:** For notational convenience we will assume that \( \kappa = 1 \); the general case can be reduced to this case by rescaling the independent variable. We first seek a solution \( w^- \) of our equation which is of the form \( w^- = \mathcal{K}_\nu \cdot z \), where \( z(r) \to 1 \) as \( r \to \infty \). Motivated by standard tricks used to solve ordinary differential equations we proceed as follows. Suppose we can find a \( C^2 \) solution \( z \) of the Volterra integral equation

\[ z(r) = 1 + \int_r^\infty dr' \ K(r, r') \ h(r') \ z(r'), \]

where

\[ K(r, r') = \left[ -\mathcal{K}_\nu(r') \right]^2 \int_r^{r'} dr'' \left[ \mathcal{K}_\nu(r'') \right]^{-2}. \]
Then $z$ satisfies the differential equation

$$-z''(r) - 2k'_v(r) [k'_v(r)]^{-1} z'(r) + h(r) z(r) = 0,$$

so that the product $w = k_v \cdot z$ satisfies our equation. It is thus of interest to find $z$; we will construct a solution of the Volterra integral equation in the standard manner, namely by proving that its Neumann series converges.

Throughout the following fix $r_0 \in \mathbb{R}_+$ such that $k_v(r) > 0$ if $r \geq r_0$. Let us first examine some properties of the kernel $K$: there is a constant $\rho$ such that

$$0 \leq K(r,r') \leq \left[ e^{r'} k'_v(r') \right]^2 \sup_{r \leq r' \leq r''} \left[ e^{r''} k'_v(r'') \right]^2 \cdot \exp \left(-2r'\right) \int_r^{r'} \exp \left(-2r''\right) dr'' \leq \frac{1}{2} \rho$$

and

$$0 \leq -\frac{\partial K}{\partial r}(r,r') = \left[ k'_v(r') \right]^2 \left[ k'_v(r) \right]^{-2} \leq \rho$$

for all $r' \geq r \geq r_0$. For later convenience let $H(r) = \int_r^\infty dr' |h(r')|$. Note that $H(r) = o(1)$ as $r \to \infty$ if $|h|$ is integrable at infinity, and that $H(r) = o(\exp(-\mu r))$ as $r \to \infty$ if $h(r) = o(\exp(-\mu r))$ as $r \to \infty$; this latter fact follows because

$$\lim_{r \to \infty} \frac{H(r)}{\exp(-\mu r)} = \lim_{r \to \infty} \frac{-\frac{1}{2} \rho |h(r)|}{-\mu \exp(-\mu r)} = 0$$

by l'Hôpital's rule.

Define $z_0 = 1$, and for non-negative integers $j$ define $z_{j+1}$ inductively by
\[ z_{j+1}(r) = \int_r^\infty dr' K(r, r') h(r') z_j(r'). \]

Then we find that the following estimates hold:

\[ |z_j(r)| \leq (j!)^{-1} [H(r)]^j \]

and

\[ \frac{1}{2} |z'_j(r)| \leq (j!)^{-1} [H(r)]^j. \]

The first estimate may be proved by induction on \( j \): \( |z_0| \leq 1 \), and if \( |z_j(r)| \leq (j!)^{-1} [H(r)]^j \) then

\[ |z_{j+1}(r)| \leq \int_r^\infty dr' \frac{1}{2} \rho \cdot |h(r')| \cdot (j!)^{-1} [H(r')]^j = [(j+1)!]^{-1} [H(r)]^{j+1}. \]

The second estimates follows from the first and the formula

\[ \bar{z}_{j+1}'(r) = \int_r^\infty dr' \frac{\partial K}{\partial r'} (r, r') h(r') z_j(r'). \]

By summing the first estimate we conclude that the series \( \sum_{j=0}^\infty z_j \) converges uniformly on \( [r_o, \infty[ \) to a function \( z \) such that

\[ |z(r) - 1| \leq \exp(H(r)) - 1 \leq \exp(H(r_o)) \cdot H(r) \]

for all \( r > r_o \); the second estimate, along with the formula

\[ \bar{z}_{j+1}''(r) = h(r) \bar{z}_{j+1}'(r) - \frac{1}{2} h(r) \left[ h(r) \left[ h(r) \right]^{-1} \right] z_j'(r), \]
shows that \( z \) is \( c^2 \) on \( ]r_0, \infty[ \) and that
\[
\frac{1}{2} \left| z'(r) \right| \leq \exp \left( H(r) \right) - 1 \leq \exp \left( H(r_0) \right) \cdot H(r)
\]
for all \( r > r_0 \). In addition, \( z \) is a solution of the Volterra integral equation, as follows by the Lebesgue dominated convergence theorem.

As a consequence \( w^- = k_y \cdot z \) is a solution of our original equation with the properties that
\[
w^-(r) = k_y(r) z(r) = k_y(r) \cdot (1 + O(H(r))) = \exp(-r) \cdot (1 + o(1))
\]
and
\[
[w^-]'(r) = \frac{d}{dr} z(r) + k_y(r) z'(r) = w^-(r) \cdot [1 + O(r^{-2}) + O(H(r))]
\]
as \( r \to \infty \). We may construct a second solution \( w^+ \) by setting
\[
w^+(r) = 2w^-(r) \int_{r_1}^r dr' \left[ w^-(r') \right]^{-2}
\]
for \( r > r_1 \) (where we choose \( r_1 > r_0 \) large enough that \( w^-(r) \neq 0 \) for all \( r \geq r_1 \)). That \( w^+ \), so defined, solves the same equation as does \( w^- \) is easily checked directly; the motivation for defining \( w^+ \) in this way comes from requiring that the Wronskian of \( w^+ \) and \( w^- \) be a constant, which is taken to be -2. The asymptotic behavior of \( w^+ \) may be determined as follows: by l'Hôpital's rule
\[
\lim_{r \to \infty} \frac{w^+(r)}{w^-(r)} = \lim_{r \to \infty} \frac{2 \int_{r_1}^r dr' \left[ w^-(r') \right]^{-2}}{\left[ w^-(r) \right]^{-2}} = \lim_{r \to \infty} \frac{2 \left[ w^-(r) \right]^2}{-2 w^-(r) w'^-(r)} = 1
\]
so that \( w^+(r) = \exp(r) \cdot (1 + o(1)) \) as \( r \to \infty \); and

\[
\lim_{r \to \infty} \left[ [w^+]'(r) [w^+(r)]^{-1}\right] = \lim_{r \to \infty} \left( [w^-]'(r) [w^-(r)]^{-1} + 2 [w^-(r) w^+(r)]^{-1} \right) = 1
\]

so that \([w^+]'(r) = w^+(r) \cdot (1 + o(1)) \) as \( r \to \infty \). This completes the proof. \( \Box \)

We are now in a position to prove

**Theorem II.5.3.** Suppose \((R,S) \in H\) is a \( C^2 \) solution of (II.3-1) such that \( F(R,S) \) is finite; let \( m = [V''(1)]^{1/2} \). Then there exist constants \( \alpha \) and \( \beta \) such that

\[
S(r) = -\alpha + \beta r K_1(r) \cdot \left( 1 + o \left( \exp \left(-\min\{m,2\}r \right) \right) \right)
\]

as \( r \to \infty \), and such that:

(a) if \( m < 2 \),

\[
R(r) = 1 - \alpha K_0(mr) \cdot \left( 1 + o \left( \exp(-mr) \right) \right)
\]

\[
- \left[ m^2 - 4 \right]^{-1} \beta^2 [K_1(r)]^2 \cdot \left( 1 + o(1) \right);
\]

(b) if \( m = 2 \),

\[
R(r) = 1 - \frac{1}{2} \beta^2 r [K_1(r)]^2 \cdot \left( 1 + o(1) \right);
\]

and

(c) if \( m > 2 \),

\[
R(r) = 1 - \left[ m^2 - 4 \right]^{-1} \beta^2 [K_1(r)]^2 \cdot \left( 1 + o(1) \right)
\]

as \( r \to \infty \).

**proof:** As mentioned above, it is convenient to define the fields \( u \) and \( v \) by
\( u(r) = r^{1/2} \left( 1 - R(r) \right) \)

and

\( v(r) = r^{-1/2} \left( \eta + S(r) \right) \)

since \( R \) and \( S \) solve (II.3-1), \( u \) and \( v \) solve (II.5-2). By assumption \( R - \check{R} \in H_{\check{R}} \); consequently \( u \) and hence \( u' \) are square-integrable at infinity, because \( u' \) is given by

\[
u'(r) = -r^{1/2} \left( R - \check{R} \right)'(r) + \frac{1}{2} r^{-1} u(r) \]

for \( r \geq 2 \). Therefore \( (u^2)' = 2u \cdot u' \) is integrable at infinity, whence

\[
\lim_{r \to \infty} u(r) \text{ exists; but this limit must vanish in order that } u^2 \text{ be integrable at infinity. Since the third term}
\]

\[
\int_0^\infty r \, dr \frac{1}{2} r^{-2} R^2(r) \left( S(r) + \eta \right)^2 = \int_0^\infty dr \frac{1}{2} \left( 1 - r^{-1/2} u(r) \right)^2 v^2(r)
\]

in \( F(R, S) \) is finite it follows that \( v \) is square-integrable at infinity. Again by assumption \( S - \check{S} \in H_S \), so that \( v' \), given by

\[
v'(r) = r^{-1/2} \left( S - \check{S} \right)'(r) - \frac{1}{2} r^{-1} v(r)
\]

for \( r \geq 2 \), is also square-integrable at infinity. As before we conclude that \( v(r) \to 0 \) as \( r \to \infty \).

An examination of (II.5-2) shows that we may apply Lemma II.5.1 to \( v \); given any positive \( \varepsilon < 1 \)

\[
v(r) = O \left( \exp \left[ - (1 - \varepsilon)^{1/2} r \right] \right)
\]
\[ u(r) = O \left( \exp \left[ - \min \left\{ m, 2 \right\} (1 - \varepsilon)^{1/2} r \right] \right) \]

as \( r \to \infty \).

Again referring to (II.5-2) we see that Lemma II.5.2 applies to the equation satisfied by \( v \). Since \( v \) vanishes at infinity it must be proportional to the subdominant solution constructed in Lemma II.5.2, so there exists a constant \( \beta' \) such that

\[ v(r) = \beta' \, k_1(r) \cdot \left( 1 + o \left( \exp \left[ - \min \left\{ m, 2 \right\} (1 - \varepsilon)^{1/2} r \right] \right) \right) \]

as \( r \to \infty \).

Consider now the equation satisfied by \( u \), which we will write as

\[ -u''(r) + \left[ m^2 - r^{-2}/4 + h(r) \right] u(r) = q(r). \]

Let \( u_+ \) and \( u_- \) denote the solutions of the corresponding homogeneous equation that are constructed in Lemma II.5.2. Using \( u_+ \) and \( u_- \) we may construct a particular solution \( u_p \) of the equation satisfied by \( u \): let

\[ u_p(r) = (2m)^{-1} \int_{r_1}^{r} \! dr' \, u_-(r) \, u_+(r') \, q(r') + (2m)^{-1} \int_{r}^{\infty} \! dr' \, u_+(r) \, u_-(r') \, q(r') \]

for \( r \geq r_1 \). Then

\[ -u_p''(r) + \left( m^2 - r^{-2}/4 + h(r) \right) u_p(r) = -(2m)^{-1} W(r) \, q(r), \]
where the Wronskian \( \bar{W}(r) = u_+(r) u'_-(r) - u_-(r) u'_+(r) = -2m(1 + o(1)) \) by Lemma II.5.2; but since \( \bar{W}' = 0 \) we must have that \( \bar{W} = -2m \), so that \( u_p \) satisfies the correct equation. Because \( u - u_p \) satisfies the homogeneous equation there exist constants \( \alpha_+ \) and \( \alpha_- \) such that \( u = \alpha_+ u_+ + \alpha_- u_- + u_p \). We will determine the asymptotic behavior of \( u \) by examining \( u_p \).

First of all,

\[
\lim_{r \to \infty} \frac{(2m)^{-1} \int_r^\infty dr' \, u_-(r') \, g(r')}{g(r) \left[u_+(r)\right]^{-1}} = \lim_{r \to \infty} \frac{-(2m)^{-1} u_-(r) \, g(r)}{g(r) \left[u_+(r)\right]^{-1} \left\{ \frac{g'(r)}{g(r)} - \frac{u'_+(r)}{u_+(r)} \right\}} = (2m)^{-1} \left[ m - \lim_{r \to \infty} \frac{g'(r)}{g(r)} \right]^{-1}
\]

by l'Hôpital's rule. But with \( g(r) = r^{-\frac{3}{2}} v^2(r) \),

\[
\frac{g'(r)}{g(r)} = -\frac{1}{2} r^{-1} + 2 \frac{v'(r)}{v(r)} = -2 \left( 1 + \frac{r^{-1}}{4} + O(r^{-2}) \right)
\]
as \( r \to \infty \) (by Lemma II.5.2), so that the second term in \( u_p \) exhibits the behavior

\[
(2m)^{-1} \int_r^\infty dr' \, u_+(r') \, u_-(r') \, g(r') = (2m)^{-1} (m+2)^{-1} \, g(r) \cdot (1 + o(1))
\]
as \( r \to \infty \).

Suppose \( m < 2 \). Then the first term in \( u_p \) may be written as

\[
(2m)^{-1} \int_r^\infty dr' \, u_+(r') \, g(r') \cdot u_-(r) = (2m)^{-1} \int_r^\infty dr' \, u_-(r) \, u_+(r') \, g(r')
\]
because \( u_+(r) \, g(r) \) vanishes exponentially as \( r \to \infty \). But
\[
\lim_{r \to \infty} \frac{-(2m)^{-1} \int_0^r \frac{d\rho}{\rho} \kappa_3(\rho) q(\rho)}{q(\rho) [\kappa(\rho)]^{-1}} = \lim_{r \to \infty} \frac{(2m)^{-1} \kappa_3(r) q(r)}{q(\rho) [\kappa(\rho)]^{-1} \left\{ \frac{q'(r)}{q(r)} - \frac{u'(r)}{u(r)} \right\}} = (2m)^{-1} (m - 2)^{-1}.
\]

Thus there is a constant \( a'' \) such that
\[
u_0(r) = a'' \nu_-(r) + [m^2 - 4]^{-1} g(r) \cdot (1 + o(1))
\]
as \( r \to \infty \). Since \( u \) vanishes at infinity it follows that there is a constant \( a' \) such that
\[
u(r) = a' \kappa_0(mr) \cdot \left( 1 + o \left( \exp \left[ -m(1 - \varepsilon)^{1/2} r \right] \right) \right)
+ [m^2 - 4]^{-1} r^{-1/2} \left[ \beta' \kappa_3(r) \right]^2 \cdot (1 + o(1))
\]
as \( r \to \infty \). If on the other hand \( m > 2 \), then because \( g \cdot [\kappa]^{-1} \) grows (exponentially) at infinity,
\[
\lim_{r \to \infty} \frac{(2m)^{-1} \int_0^r \frac{d\rho}{\rho} \kappa_3(\rho) q(\rho)}{q(\rho) [\kappa(\rho)]^{-1}} = \lim_{r \to \infty} \frac{(2m)^{-1} \kappa_3(r) q(r)}{q(\rho) [\kappa(\rho)]^{-1} \left\{ \frac{q'(r)}{q(r)} - \frac{u'(r)}{u(r)} \right\}} = (2m)^{-1} (m - 2)^{-1},
\]
so that
\[
u(r) = a_- \kappa_0(mr) \cdot \left( 1 + o \left( \exp \left[ -2(1 - \varepsilon)^{1/2} r \right] \right) \right)
+ [m^2 - 4]^{-1} r^{-1/2} \left[ \beta' \kappa_3(r) \right]^2 \cdot (1 + o(1)).
\]
Lastly, in case \( m = 2 \), \( r \ g(r)\{u_-(r)\}^{-1} \) grows (as \( r^{1/2} \)) as \( r \to \infty \), so

\[
\lim_{r \to \infty} \frac{(2m)^{-1} \int_{r_0}^{r} dr' \ u_+(r') \ g(r')}{r \ g(r) \ \left[ u_-(r) \right]^{-1}} = \lim_{r \to \infty} \frac{(2m)^{-1} \ u_+(r) \ \frac{g(r)}{r}}{r \ g(r) \ \left[ u_-(r) \right]^{-1} \ \left( r^{-1} + \frac{g(r)}{g(r)} \frac{u_+(r)}{u_-(r)} \right)}
\]

\[
= (2m)^{-1} \lim_{r \to \infty} \left[ r \left( \frac{1}{2} r^{-1} + O(r^{-2}) \right) \right]^{-1} = m^{-1}
\]

because \( g'(r)/g(r) = -2 - \frac{1}{2} r^{-1} + O(r^{-2}) \) as \( r \to \infty \); thus

\[
u(r) = \alpha_+ k_0(mr) \cdot \left( 1 + o \left( \exp \left[ -m \left( 1 - \varepsilon \right)^{1/2} r \right] \right) \right)
\]

\[
+ m^{-1} \ r^{1/2} \ \left[ \beta' k_1(r) \right]^2 \ \left( 1 + o(1) \right)
\]

as \( r \to \infty \).

Recycling these formulae through Lemma II.5.2 shows that the \( (1 - \varepsilon)^{1/2} \) may be replaced by \( 1 \). Finally, application of the definitions of \( u \) and \( v \) in terms of \( R \) and \( S \) and of \( k_{\psi} \) in terms of \( K_{\psi} \) finishes the proof. []

Let us make some remarks about the constants \( \alpha \) and \( \beta \) appearing in Theorem II.5.3. First of all, \( \beta \) must be non-zero because \( v \) cannot vanish identically; the sign of \( \beta \) may be determined as follows. Note that

\[
\frac{d}{dr} \left( v(r) v'(r) \right) = \left[ \left( 1 + r^{-1/2} u(r) \right)^2 + 3r^{-2}/4 \right] v^2(r) + [v'(r)]^2
\]

so that \( v \cdot v' \) is strictly increasing; but \( \lim_{r \to \infty} v(r) v'(r) = 0 \), so we see that \( v(r) v'(r) < 0 \) for all \( r \). Since \( v(r) \) is positive for small enough \( r \) we conclude that \( v(r) \) is positive for all \( r \), whence \( \beta \) is positive.

(This argument also shows that \( v \), and hence \( S \), is monotonically decreasing,
just as our variational solution is; cf. the use of this argument in [W].

It seems that we cannot argue in this way to show that \( \alpha \) is positive, but under the assumption that \( R(r) \leq 1 \) for large \( r \) it is clear that this is so. In this case we have completely determined the leading asymptotic behavior of \( R \) and \( S \).

Using a different approach we may derive inequalities which \( \alpha \) and \( \beta \) satisfy in certain circumstances. Define the function \( i_\nu \) by

\[
i_\nu(r) = (2\pi r)^{\nu/2} I_\nu(r),
\]

where \( I_\nu \) is the usual [BMP] modified Bessel function of order \( \nu \) that is subdominant at the origin; \( i_\nu \) satisfies the same equation as does \( k_\nu \).

We will need to know the following properties of \( i_\nu \): if \( \nu^2 \) is real then

\[
i_\nu(r) = e^{-r} (1 + O(r^{-1}))
\]

and

\[
i_\nu'(r) = i_\nu(r) (1 + O(r^{-2}))
\]

as \( r \to \infty \), and

\[
i_\nu(r) = 2 \pi^{\nu/2} [\Gamma(\nu+1)]^{-1} \left( \frac{r}{2} \right)^{\nu+\frac{1}{2}} (1 + O(r))
\]

and

\[
i_\nu'(r) = \pi^{\nu/2} [\Gamma(\nu+1)]^{-1} (\nu + \frac{1}{2}) \left( \frac{r}{2} \right)^{\nu-\frac{1}{2}} (1 + O(r))
\]
as \( r \to 0 \). (See Appendix II.)

If we write the equation satisfied by \( v \) in the form

\[
-u''(r) + \left[ 1 + 3r^{-1/4} \right] u(r) = (1 - R^2(r))^{-1/2} (v + S(r)) = \frac{f_v}{u}(r)
\]

we find that \( [i_y'v - i_z'v']' = i_x f_v \). By the proofs of Lemma II.5.2 and Theorem II.5.3, the function \( z \) defined by \( v = \beta' k_1 \) has the properties that

\[
z(r) = 1 + o(1)
\]

and

\[
z'(r) = o(1)
\]

as \( r \to \infty \), so that

\[
i_y'(r) u(r) - i_z'(r) v'(r) = \beta' \left[ i_y'(r) k_1(r) - i_z'(r) k_1'(r) \right] z(r) - \beta' i_x'(r) k_1(r) z(r)
\]

\[
= 2 \beta' + o(1) = (2\pi)^{1/2} \beta + o(1)
\]

as \( r \to \infty \). On the other hand

\[
i_y'(r) u(r) - i_z'(r) v'(r) = \left[ i_y'(r) r^{-1/2} + \frac{1}{2} s'(r) r^{-1/2} \right] (v + S(r)) - i_z'(r) r^{1/2} S(r)
\]

\[
= (2\pi)^{1/2} v + o(1)
\]

as \( r \to 0 \) by the proof of Theorem II.4.4. Therefore
\[(2\pi)^{\nu_{\mathcal{K}}} (\beta - n) = \int_0^\infty dr \; i_1(r) f_\nu(r)\]

Since \(f_\nu > 0\) if \(|R| \leq 1\) we find in particular that

\[\beta > n\]

under the assumption that \(|R| \leq 1\).

In a similar fashion we may write

\[-u''(r) + \left[ m^2 - r^{-2/\ell} \right] u(r) = r^{\nu_2} \left[ V'(R(r)) + V'(1-R(r)) + r^{-2} \left( S(r) + n \right)^2 R(r) \right] = f_u(r)\]

and show that

\[(2\pi)^{\nu_2} m \alpha = 2m' \alpha' = \left[ m i_1' (mr) u(r) - i_1(mr) u'(r) \right]_{0}^{\infty} = \int_0^\infty dr \; i_1(mr) f_u(r)\]

so long as \(m < 2\). Therefore

\[\alpha > 0\]

in case \(R \geq 0\) and the potential \(V\) satisfies

\[V'(R) + V''(1) (1 - R) \geq 0\]

for all \(R \geq 0\). For example, the quartic double-well potential \(V_m\) defined by

\[V_m(R) = \frac{m^2}{8} (1 - R^2)^2\]

satisfies this condition.
III.1 The Yang-Mills Higgs Model

The Yang-Mills Higgs model describes a scalar Higgs field which is self-coupled via a potential $V$ and which interacts with a Yang-Mills (i.e. $SU(2)$) gauge field in 3-dimensional Euclidean space. The Higgs field transforms according to some finite dimensional, real, symmetric representation $g \mapsto U(g) : W \to W$ of the Lie group $SU(2)$ in the vector space $W$, and thus $\phi$ is a $W$-valued function on $\mathbb{R}^3$; the gauge field is a 1-form on $\mathbb{R}^3$ taking values in the Lie algebra $su(2)$ of $SU(2)$. The potential $V$ is assumed to be twice continuously differentiable, non-negative, and symmetric about the origin, and to have a zero at $R_0 > 0$ with $V''(R_0) > 0$. Using units in which $R_0 = 1$ and $e/\hbar c = 1$ (where $e$ is the coupling constant for the interaction between the Higgs and gauge fields) the Euclidean action for the Yang-Mills higgs model is

$$\mathcal{F}(\phi, A) = \int_{\mathbb{R}^3} d^3x \left\{ \frac{1}{2} \| F(A) \|^2 + \frac{1}{2} \| d_A \phi \|^2 + V(\| \phi \|) \right\},$$

where $F(A) = dA + \frac{1}{2}[A, A]$ is the field strength (curvature) of $A$, and $d_A \phi = d\phi + U(A) \phi$ is the covariant derivative of $\phi$. The critical points of $\mathcal{F}$ formally satisfy the equations

$$d_A^* d_A \phi + V'(\| \phi \|) \frac{\phi}{\| \phi \|} = 0$$

and

$$d_A^* F(A) + J_A(\phi) = 0;$$

here the Higgs current $J_A(\phi)$ is defined so that
\[
\left( X \mid J_{A}(\phi) \right)_{SU(2)} = \left( \bigcup (X) \phi \mid d_{A} \phi \right)_{W}
\]

for all \( X \in su(2) \).

We wish to construct so-called monopole solutions of (III.1-1), characterized by having a non-zero monopole number \( n \), which is an integer. The monopole number, which is a topological invariant of the gauge field, is defined as follows: the field \( f = *|d|_{X} \wedge F(A)|/||d|_{X} \Lambda F(A)|| \) of unit vectors in \( su(2) = \mathbb{R}^{3} \) gives rise to a map

\[
\omega \longmapsto \lim_{r \to \infty} f(r \omega)
\]

from \( S^{2} \) to \( S^{2} \); the Brouwer degree of this map is the monopole number. If the Higgs field transforms according to the adjoint representation then the monopole number may be expressed more simply as

\[
n = \frac{1}{4\pi} \int_{\mathbb{R}^{3}} \left( d_{A} \phi \mid F(A) \right)_{SU(2)} .
\]

(There is an implied wedge-product in the integrand here.) The Bianchi identity \( d_{A} F(A) = 0 \) shows that \( d(\phi \mid F(A))_{su(2)} = (d_{A} \phi \mid F(A))_{su(2)} \), so we also have that

\[
n = \frac{1}{4\pi} \lim_{r \to \infty} \int_{\partial B(0, r)} \left( \phi \mid F(A) \right)_{su(2)}
\]

by Stokes' theorem.

Let us remark here that in case \( \phi \) transforms according to the adjoint representation there is the following interpretation of the monopole number in terms of the topological charge (i.e. second Chern number) of an auxiliary gauge field. Consider the manifold \( \mathbb{R}^{3} \times S^{1} \); let \( \chi \) denote the angular
coordinate for the $S^1$ factor. Regarding $\phi$ and $A$ as $\chi$-independent
fields on $\mathbb{R}^3 \times S^1$ define the gauge field $\hat{A}$ on $\mathbb{R}^3 \times S^1$ by

$$\hat{A} = A + \phi \, d\chi.$$ 

Then

$$F(\hat{A}) = F(A) + \frac{1}{A} \phi \wedge d\chi,$$

so that the second Chern class for $\hat{A}$ is

$$\left(\frac{\text{i} \, \pi^3}{4 \pi^2} \right)^{-1} \left( F(A) \, | \, F(A) \right)_{SU(2)} = (4\pi)^{-1} \left( \frac{1}{A} \phi \, | \, F(A) \right)_{SU(2)} \wedge (2\pi)^{-1} d\chi.$$

Thus the second Chern number of $\hat{A}$, given by integrating the second Chern
class over $\mathbb{R}^3 \times S^1$, is the monopole number. (This description is
reminiscent of, but distinct from, the interpretation of the monopole number
of time-periodic solutions of the Yang-Mills Higgs model in 4-dimensional
Minkowski space in terms of a topological charge [CJ].)

We look for isotropic solutions of (III.1-i) which have the form

$$\phi(x) = R(|x|) \, \hat{\phi}(x)$$

and

$$A(x) = S(|x|) \, \hat{A}(x),$$

where $\hat{\phi}$ and $\hat{A}$ are obtained from fields on the unit sphere $S^2$ in $\mathbb{R}^3$
by pullback under the map $x \mapsto x/|x|$ from $\mathbb{R}^3$ to $S^2$ (i.e. they depend
only on angles); we normalize $\dot{\phi}$ and $\dot{A}$ by requiring that

$$R(r) \rightarrow 1$$

and

$$\mathcal{S}(r) \rightarrow -1$$

as $r \to \infty$. Michel, O'Raifeartaigh, and Rawnsley [MO'RR] have argued that the assumption that the Euclidean action $\mathcal{F}(\phi, A)$ is finite forces one to take

$$\dot{\phi}(x) = \sum_m \tau^m \gamma^\ell_m \left( \frac{x}{|x|} \right)$$

and

$$\dot{A}(x) = \sum_{a, j, k} \frac{\sigma^a}{2i} \varepsilon_{ajk} \frac{x^j dx^k}{|x|^3}$$

for some integer $\ell$, where the $\sigma^a$, $a = 1, 2, 3$, are the usual Pauli spin matrices, and the $\tau^m$, $m = -\ell, ... , \ell$, transform according to the $\ell$th irreducible representation of SU(2) in such a way that

$$\frac{1}{i} L^a \phi + U \left( \frac{\sigma^a}{2i} \right) \dot{\phi} = 0; \quad (III.1)$$

(here $L^a = \varepsilon^{ajk} x_j \frac{1}{i} \frac{\partial}{\partial x^k}$ is the usual angular momentum operator). For example, we may take

$$\dot{\phi}(x) = \sum_a \frac{\sigma^a}{2i} \frac{x^a}{|x|}$$
if \( \phi \) transforms according to the \( l = 1 \) adjoint representation, as first shown by t'Hooft [t'H] and Polyakov [Py].

Let us show that if \( R \) and \( S \) satisfy the equations

\[
-R''(r) - 2r^{-1} R'(r) + \lambda(l+1) r^{-2} (S(r)+1)^2 R(r) + \nabla'(R(r)) = 0
\]

and

\[
-S''(r) + \frac{1}{2} \lambda(l+1) R^2(r) (S(r)+1) + r^{-2} S(r) (S(r)+1) (S(r)+2) = 0
\]  

(III.1-5)

then \( \phi \) and \( A \), as given by (III.1-2) and (III.1-3), satisfy (III.1-1) away from \( x = 0 \). First note that (III.1-4) implies that

\[
\tilde{D} \phi \quad = \quad U(\hat{A}) \hat{\phi}
\]

as is seen by multiplying both sides of (III.1-4) by \( \sum_{j,k} \epsilon_{ajk} \frac{x^j dx^k}{|x|^2} \) for summation on the index \( a \); second, straightforward calculations show that

\[
\tilde{D} \hat{A} \quad = \quad [\hat{A}, \hat{A}]
\]

and

\[
\tilde{D}^* \hat{A} \quad = \quad 0
\]  

(so that \( A \) satisfies the Lorentz gauge condition \( d^* A = 0 \)). We then have that

\[
\frac{d}{dA} \phi(x) \quad = \quad R'(ix) \int_{|x|} \dot{\phi}(x) + R(|x|) (S(|x|)+1) U(\hat{A}) \hat{\phi}(x)
\]
whence
\[ d^\alpha d_A \phi(x) = -x^{-1} d_A \Phi(x) \]
\[ = - (R''(|x|) + 2 |x|^{-1} R'(|x|)) d|x| \phi(x) \]
\[ = R(|x|) (S''(|x|) + 1) x^{-1} \left[ U(x^\alpha) \Lambda U(\hat{\alpha}) + S(|x|) U(\tilde{\alpha}) \Lambda U(\tilde{\alpha^\prime}) \right] \phi(x) \]
\[ = \left[ - R''(|x|) - 2 |x|^{-1} R'(|x|) + \lambda (l+1) |x|^{-1} \left( S(|x|) + 1 \right) R(|x|) \right] \phi(x) \]

because
\[ x^{-1} U(x^\alpha) \Lambda U(\hat{\alpha}) \phi(x) = x^{-1} U(x^\alpha) \Lambda U(\hat{\alpha}) \phi(x) \]
\[ = |x|^{-2} \sum_{a, b} \left( \delta_{a, b} - \frac{x_a x_b}{|x|^2} \right) U(\frac{\sigma_a^b}{x}) U(\frac{\sigma_b^a}{x}) \phi(x) \]
\[ = |x|^{-2} \sum_{a, b} \left( \delta_{a, b} - \frac{x_a x_b}{|x|^2} \right) \frac{1}{l} L^a \frac{1}{l} L^b \phi(x) \]
\[ = - |x|^{-2} \sum_{a} L^a L^a \phi(x) = - \lambda (l+1) |x|^{-2} \phi(x). \]

Moreover \( ||\phi(x)|| \) is independent of \( x \) since \( U \) is a symmetric representation, so we must have that \( ||\phi|| = 1 \) in order that \( ||\phi(x)|| \to R_0 = 1 \) as \( |x| \to \infty \); thus
\[ V' \left( ||\phi(x)|| \right) \frac{\phi(x)}{||\phi(x)||} = V' \left( R(|x|) \right) \phi(x). \]

Similarly,
\[ F(A)(x) = \left( J^A + \frac{1}{2} [A,A] \right)(x) = S(1x1) \frac{1}{2} x A(x) + S(1x1)(\frac{1}{2} S(1x1) + i) J^A(x) \]

\[ \frac{d}{dA} F(A)(x) = \ast^{-1} d \ast \ F(A)(x) = - S''(1x1) A(x) \ast S(1x1) \left( \frac{1}{2} S(1x1) + i \right) \ast^{-1} \ast \ A(x) \]

\[ + S^2(1x1) \left( \frac{1}{2} S(1x1) + i \right) \ast^{-1} \left[ \ast A, \ast \ast A \right] x \]

\[ = \left[ - S''(1x1) + \frac{1}{2} x A(x) + S(1x1) \left( S(1x1) + i \right) \left( S(1x1) + 2 \right) \right] \ast \]

because \( \ast^{-1} d \ast A(x) = - \Delta A(x) = 2 \vert x \vert^{-2} A(x) \) and \( \ast^{-1} [A, \ast dA](x) = 2 \vert x \vert^{-2} A(x) \), as is easily checked. To calculate \( J_A(q) \), consider the matrix field \( M \) defined by

\[ M^{ab}(x) = \begin{pmatrix} U \left( \frac{x^a}{\xi} \right) \phi(x) & U \left( \frac{x^b}{\xi} \right) \phi(x) \end{pmatrix} \]

From (III.1-4) and the invariance of the inner product in \( W \) it follows that the components of \( M \) transform according to the tensor product of two adjoint representations of \( SU(2) \); therefore

\[ M^{ab}(x) = \alpha (\delta^{ab} - \frac{1}{3} \frac{x^a x^b}{\vert x \vert^2}) + \beta \sum_c \epsilon^{abc} \frac{x^c}{\vert x \vert} + \gamma \delta^{ab} \]

for some choice of constants \( \alpha, \beta, \text{ and } \gamma \), by the Wigner-Eckhart theorem. But \( M \) is symmetric, and

\[ \sum_b \sum_i M^{bi}(x) \frac{x^i}{\vert x \vert} = \left( U \left( \frac{x^a}{\xi} \right) \phi(x) \right) \left( \sum_b \frac{x^b}{\vert x \vert} \right) \left( \sum_i \frac{x^i}{\vert x \vert} \right) \phi(x) \]

and
\[
\sum' M^{a_b}(x) = - \sum' \left( \dot{\phi}(x) \mid U \left( \frac{\sigma^a}{2i} \right) \cdot \frac{i}{\lambda} L^c \phi(x) \right)_W
\]
\[
= \left( \dot{\phi}(x) \mid \sum' L^c \phi(x) \right)_W = \lambda (\xi + 1)
\]

so that

\[
M^{a_b}(x) = \frac{1}{2} \lambda (\xi + 1) \left( \delta^{a_b} - \frac{x^a x^b}{|x|^2} \right).
\]

Therefore

\[
J_A(\phi)_{\alpha} = \sum' \frac{\sigma^a}{2i} \left( \frac{\sigma^a}{2i} \mid J_A(\phi) \right)_W_{\alpha}^{\beta} (x)
\]
\[
= \sum' \frac{\sigma^a}{2i} \left( U \left( \frac{\sigma^a}{2i} \right) \phi \mid A\phi \right)_W (x)
\]
\[
= R^2(|x|) \left( \frac{1}{2} \left( \frac{\sigma^a}{2i} \mid \dot{\phi}(x) \right)_W \left( U \left( \frac{\sigma^a}{2i} \right) \phi(x) \right)_W \right)
\]
\[
+ R'(|x|) R(|x|) d|x| \sum_a \frac{\sigma^a}{2i} \left( \dot{\phi}(x) \mid U \left( \frac{\sigma^a}{2i} \right) \phi(x) \right)_W
\]
\[
= \frac{1}{2} \lambda (\xi + 1) \left( \frac{1}{2} \dot{\phi}(x) \mid U \left( \frac{\sigma^a}{2i} \right) \phi(x) \right)_W
\]

because \( \sum b \frac{\sigma^b}{2i} \frac{x^b}{|x|} \mid \dot{A} \rangle = 0 \) (by direct calculation) and \( \langle \dot{\phi} \mid U \left( \frac{\sigma^a}{2i} \right) \phi \rangle = 0 \)

(by the skew-symmetry of \( U \left( \frac{\sigma^a}{2i} \right) \)). With this we finally conclude that \( \phi \)

and \( A \) satisfy (III.1-1) if \( R \) and \( S \) satisfy (III.1-5).

To calculate the monopole number of fields of the form (III.1-2) and

(III.1-3) we note that the monopole number is independent of the representation

under which the Higgs field transforms, so we may as well assume that \( \xi = 1 \).
But then

\[ n = \frac{1}{4\pi} \lim_{r \to \infty} \int_{\partial B(o, r)} R(x) \left( \frac{\phi(x)}{S(x)^{2}} \right) S(x) \, \hat{A}(x) + \hat{A}(x) \right) \frac{1}{2} S(x) + 1 \right) \frac{1}{4\pi} \int_{\partial B(o, r)} \left( \frac{\phi(x)}{S(x)^{2}} \right) \, d\hat{A}(x) \right) \]

Thus the only finite action monopoles which have the form (III.1-2) are ones with unit monopole number [MO1RR].

The critical points of the function \( F \) defined by \( F(R, S) = (4\pi)^{-1} F(\phi, A) \), so that

\[ F(R, S) = \int_{0}^{\infty} r^2 \, dr \left\{ \frac{1}{2} [R'(r)]^2 + r^{-2} [S'(r)]^2 + \frac{1}{2} \sqrt{1+k} (S(r) + 1) \frac{1}{2} \right\} R^2(r) \]

\[ + \frac{1}{2} r^{-4} S^2(r) (S(r) + 2) + V(\phi(r)) \right\} \]

formally satisfy (III.1-5) and the boundary conditions

\[ R(r) \to 0 \quad \text{and} \quad S(r) \to 0 \quad \text{as} \quad r \to 0 \]

and

\[ R(r) \to 1 \quad \text{and} \quad S(r) \to -1 \quad \text{as} \quad r \to \infty \]

In the following we will prove the existence of solutions of (III.1-1) which take the form (III.1-2) and (III.1-3) by proceeding as we did in studying the Abelian Higgs model.
The Yang-Mills Higgs Model - Existence of Solutions

In the previous section we reduced the problem of finding isotropic solutions of the Yang-Mills Higgs model to finding real-valued functions $\mathcal{R}$ and $\mathcal{S}$ on the half-line $\mathbb{R}_+$ that solve the non-linear elliptic boundary-value problem

\[
\begin{align*}
- \mathcal{R}''(r) &= 2r^{-1} \mathcal{R}'(r) + \mathcal{L}(1+1) r^{2} (\mathcal{S}(r)+1)^2 \mathcal{R}(r) + \mathcal{V}'(\mathcal{R}(r)) = 0, \\
- \mathcal{S}''(r) &= -\frac{1}{2} \mathcal{L}(1+1) \mathcal{R}^2(r) (\mathcal{S}(r)+1) + r^{2} \mathcal{S}(r)(\mathcal{S}(r)+1)(\mathcal{S}(r)+2) = 0,
\end{align*}
\]

(III.2-1)

\[
\mathcal{R}(r) \to 0 \quad \text{and} \quad \mathcal{S}(r) \to 0 \quad \text{as} \quad r \to 0, \quad (III.2-2)
\]

and

\[
\mathcal{R}(r) \to 1 \quad \text{and} \quad \mathcal{S}(r) \to -1 \quad \text{as} \quad r \to \infty. \quad (III.2-3)
\]

These equations are formally satisfied by any pair $(\mathcal{R}_{\min}, \mathcal{S}_{\min})$ that minimizes the function $F$, given by

\[
F(\mathcal{R}, \mathcal{S}) = \int_{\nu}^{v} r^2 \, dr \left\{ \frac{1}{2} \left[ \mathcal{R}'(r) \right]^2 + r^{-2} \left[ \mathcal{S}'(r) \right]^2 + \frac{1}{2} \mathcal{L}(1+1)(\mathcal{S}(r)+1)^2 \mathcal{R}^2(r) \right. \\
+ \frac{1}{2} r^{-4} \mathcal{S}^2(r)(\mathcal{S}(r)+2) + \mathcal{V}(\mathcal{R}(r)) \left. \right\}. \quad (III.2-4)
\]

Tyupkin, Fateev, and Schwartz [TFS] (in the case when $\mathcal{V}$ is a quartic double-well potential and $\mathcal{L} = 1$), along with Ramosley [RL] and [W] (for more general potentials $\mathcal{V}$ and all $\mathcal{L}$) have supplied proofs that $F$, as
a function on a space $H$ of pairs $(R, S)$, attains its infimum at some point $(R_{\text{min}}, S_{\text{min}})$ in $H$. However they do not give a proof that such minima are weak solutions of (III.2-1), a proof which involves a nontrivial argument.

In the following we present a complete proof, using a slightly different space $H$, which parallels the corresponding proof for the Abelian Higgs model. We will be brief because this section is so similar to II.3.

As before let $\tilde{R}$ be a $C^\infty$ function on $\mathbb{R}_+$ such that $0 \leq \tilde{R}(r) \leq 1$ for all $r \in \mathbb{R}_+$, $\tilde{R}(r) = 0$ if $r \leq 1$, and $\tilde{R}(r) = 1$ if $r \geq 2$; and let $\dot{\tilde{S}}(r) = -\tilde{R}(r)$ for all $r \in \mathbb{R}_+$. Write $R = \tilde{R} + \tilde{R}$ and $S = \tilde{S} + \tilde{S}$.

We will take $\tilde{R}$ to belong to the space $H_{\tilde{R}}$ which consists of those continuous functions $\tilde{R}$ on $\mathbb{R}_+$ for which the function $r^{1/2} \tilde{R}: r \mapsto r^{1/2} \tilde{R}(r)$ is bounded and for which the distributional derivative $\tilde{R}'$ belongs to $L^2(\mathbb{R}_+, r^2 \, dr)$; $H_{\tilde{R}}$ is an inner product space when equipped with the inner product induced by the norm $\| \cdot \|_{\tilde{R}}$ given by

$$\| \tilde{R} \|_{\tilde{R}}^2 = \int_0^\infty r^2 \, dr \left[ \tilde{R}'(r) \right]^2,$$

and we show below that $H_{\tilde{R}}$ is a Hilbert space. We will take $\tilde{S}$ to belong to the space $H_{\tilde{S}}$ which consists of those continuous functions $\tilde{S}$ on $\mathbb{R}_+$ for which the function $r^{-1/2} \tilde{S}: r \mapsto r^{-1/2} \tilde{S}(r)$ is bounded and for which $\tilde{S}'$ belongs to $L^2(\mathbb{R}_+, dr)$; we equip $H_{\tilde{S}}$ with the inner product induced by the norm $\| \cdot \|_{\tilde{S}}$ given by

$$\| \tilde{S} \|_{\tilde{S}}^2 = \int_0^\infty \, dr \left[ \tilde{S}'(r) \right]^2,$$

and will show that $H_{\tilde{S}}$ is a Hilbert space. Define $H$ to consist of pairs $(R, S)$ with $R = \tilde{R} + \tilde{R}$ for some $\tilde{R}$ in $H_{\tilde{R}}$ and with $S = \tilde{S} + \tilde{S}$ for some $\tilde{S}$ in $H_{\tilde{S}}$, and define the function $F$ on $H$ by (III.2-4).
Lemma III.2.1. If \( R \in H_R \), then

\[
\sup_{r \in R_+} r^{1/2} \left| \frac{d}{dr} R(r) \right| \leq \| R \|_R.
\]

If \( S \in H_S \), then

\[
\sup_{r \in R_+} r^{1/2} \left| \frac{d}{dr} S(r) \right| \leq \| S \|_S.
\]

**proof:** Let \( r \in R_+ \). Since \( R(r) = O(r^{1/2}) \) as \( r \to \infty \),

\[
\tilde{R}(r) = - \int_r^\infty \frac{d}{dr'} \tilde{R}'(r') dr',
\]

so that

\[
|\tilde{R}(r)| \leq \int_r^\infty \frac{d}{dr'} |\tilde{R}'(r')| \leq \left[ \int_r^\infty dr' \left( r' \right)^{-1} \right]^{1/2} \left[ \int_r^\infty dr' \left[ \tilde{R}'(r') \right]^2 \right]^{1/2}
\]

by the Schwarz inequality. Similarly

\[
|\tilde{S}(r)| \leq \int_r^0 \frac{d}{dr'} |\tilde{S}'(r')| \leq \left[ \int_r^0 dr' \left( r' \right)^{-1} \right]^{1/2} \left[ \int_r^0 dr' \left[ \tilde{S}'(r') \right]^2 \right]^{1/2}. \]

Lemma III.2.2. \( H_R \) and \( H_S \) are Hilbert spaces.

**proof:** The proof mimics the proof of Lemma II.3.2 exactly, so we omit it. \( \square \)

Lemma III.2.3. Suppose \( (R, S) \in H \). Define \( R^{\text{mod}} \) to be the function

\[
R \mapsto \min \left\{ |R(r)|, 1 \right\}
\]

Then \( (R^{\text{mod}}, S) \in H \) and \( F(R^{\text{mod}}, S) \leq F(R, S) \).

**proof:** The proof is essentially the same as the proof of Lemma II.3.3. \( \square \)
After these preliminaries we can prove

**Theorem III.2.4.** There exists a pair $(R_{\min}, S_{\min})$ in $H$ such that

$$F(R_{\min}, S_{\min}) = \inf_{(R, S) \in H} F(R, S)$$

and such that $0 \leq R \leq 1$.

**Proof:** Choose a sequence $j \mapsto (R_j, S_j)$ in $H$ such that

$$\lim_{j \to \infty} F(R_j, S_j) = \inf_{(R, S) \in H} F(R, S)$$

and such that $F(R_j, S_j)$ is finite for each $j$. We may assume that $0 \leq R_j \leq 1$ by invoking Lemma III.2.3. By the triangle inequality,

$$F(R_j, S_j) \geq \frac{1}{2} \left( \{ \int_0^\infty r^2 dr \left[ \tilde{R}_j(r) \right]^2 \}^{1/2} - \{ \int_0^\infty r^2 dr \left[ \tilde{R}'(r) \right]^2 \}^{1/2} \right)^2$$

$$+ \left( \{ \int_0^\infty dr \left[ \tilde{S}_j'(r) \right]^2 \}^{1/2} - \{ \int_0^\infty dr \left[ \tilde{S}'(r) \right]^2 \}^{1/2} \right)^2,$$

where we have written $R_j = \tilde{R} + \tilde{R}_j$ and $S_j = \tilde{S} + \tilde{S}_j$. This implies that the $\tilde{R}_j$ and $\tilde{S}_j$ lie inside closed balls in $H_R$ and $H_S$, respectively. By the Banach-Alaoglu theorem we may assume that $\tilde{R}_j \rightharpoonup \overline{R}$ and $\tilde{S}_j \rightharpoonup \overline{S}$ weakly in $H_R$ and $H_S$, respectively, for some $\overline{R} \in H_R$ and $\overline{S} \in H_S$. Set $R_{\min} = \overline{R} + \overline{R}_\infty$ and $S_{\min} = \overline{S} + \overline{S}_\infty$.

In exactly the same way as in the proof of Theorem II.3.5, we may use Lemma II.3.4 to construct subsequences of the nets $j \mapsto R_j$ and $j \mapsto S_j$ such that the convergence to $\overline{R}$ and $\overline{S}$ is uniform over any compact subset of $[R_+ \mapsto$. In particular $0 \leq R_{\min} \leq 1$. By Fatou's lemma
\[
\int_0^\infty r^2 \, dr \left\{ \frac{1}{2} l(l+1) r^{-2} (S_{\min}(r)+1)^2 R_{\min}^2(r) + \frac{1}{4} r^{-4} S_{\min}^2(r) \left( S_{\min}(r) + 2 \right)^2 + V(R_{\min}(r)) \right\} \\
\leq \liminf_{r \to \infty} \int_0^\infty r^2 \, dr \left\{ \frac{1}{2} l(l+1) r^{-2} \left( S(r)+1 \right)^2 R_j^2(r) + \frac{1}{2} r^{-4} S_j^2(r) \left( S_j(r) + 2 \right)^2 + V(R_j(r)) \right\}
\]

Furthermore the maps

\[
\tilde{R} \mapsto \int_0^\infty r \, dr \cdot \frac{1}{2} \left[ \tilde{R}'(r) + \tilde{R}''(r) \right]^2
\]

and

\[
\tilde{S} \mapsto \int_0^\infty r \, dr \cdot r^{-2} \left[ \tilde{S}'(r) + \tilde{S}''(r) \right]^2
\]

are strongly continuous and convex, hence weakly lower semicontinuous.

Therefore

\[
F(R_{\min}, S_{\min}) = \inf_{(R,S) \in \mathcal{H}} F(R,S) \quad \square
\]

We now prove that minima of \( F \) are weak solutions of (III.2-1) in the sense of Definition III.2.5. An element \((R,S)\) of \( \mathcal{H} \) with \( F(R,S) \) finite is said to be a weak solution of (III.2-1) if and only if

\[
\int_0^\infty r \, dr \cdot R'(r) \tilde{R}'(r) + \int_0^\infty r^2 \, dr \left[ l(l+1) r^{-2} (S(r)+1)^2 R(r) + V'(R(r)) \right] \tilde{R}(r) = 0
\]

and

\[
\int_0^\infty r \, dr \cdot r^{-2} S(r) \tilde{S}'(r) + \int_0^\infty r^2 \, dr \left[ \frac{1}{2} l(l+1) r^{-2} R^2(r) (S(r)+1) + r^4 S(r)(S(r)+1)(S(r)+2) \right] \tilde{S}(r) = 0
\]
for all $R \in H_R$ and $S \in H_S$ such that $\bar{R}(r)$ and $\bar{S}(r)$ vanish for all sufficiently large $r$, $\bar{R}(r) = O(r)$ as $r \to 0$, and $\bar{S}(r) = O(r^2)$ as $r \to 0$.

**Theorem III.2.6.** A minimum $(R,S)$ of $F$ over $H$ is a weak solution of (III.2.1).

**proof:** As in the proof of Theorem II.3.7 we see that $|R| \leq 1$ because $(R,S)$ minimizes $F$. Let $t \in \mathbb{R}_+$ and let $R$ and $S$ be as in Definition III.2.5. Then

$$F(R + t\bar{R}, S) = F(R, S) + t \int_0^\infty r^2 dr \bar{R}(r)\bar{R}'(r) + \frac{1}{2} \left[ r^2 \int_0^\infty S(r) \bar{S}(r) dr \right] + \left( \left( S(\infty) + \frac{1}{2} \left( S' \right)^2 \right) \right),$$

where $V(t,r) = V(R(r) + t\bar{R}(r)) - [V(R(r)) + tV'(R(r))\bar{R}(r)]$. Note that $|V(t,r)| \leq \text{const. } t^2 |\bar{R}|(r)$ by Taylor's theorem and the fact that $|R| \leq 1$.

Similarly,

$$F(R, S + t\bar{S}) = F(R, S) + 2t \int_0^\infty r^2 dr \bar{R}(r) \bar{S}'(r) S'(r)$$

$$\quad + 2t \left( \left( S(\infty) + \frac{1}{2} \left( S' \right)^2 \right) \right) + \frac{1}{2} \left( S(\infty) + \frac{1}{2} \left( S' \right)^2 \right) + \frac{1}{2} \left( S(\infty) + \frac{1}{2} \left( S' \right)^2 \right) + \frac{1}{2} \left( S(\infty) + \frac{1}{2} \left( S' \right)^2 \right),$$

Therefore $\frac{d}{dt} F(R + tR, S)$ and $\frac{d}{dt} F(R, S + tS)$ exist and must vanish, whence $(R, S)$ is a weak solution of (III.2.1).

Let us compare the above proof to the corresponding proof for the Abelian Higgs model. In the case of the Abelian Higgs model it is necessary...
to require that \( V(R_1) \) not vanish for \( |R_1| < 1 \) in order that the finiteness of \( \int_0^\infty r dr \, V(R(r)) \) control \( R(r) \) at large \( r \). Technically this condition arises as follows. The completion of \( C_\infty_c(\mathbb{R}_+) \) in the norm

\[
\tilde{\mathcal{H}} \mapsto \left\{ \int_0^\infty r dr \left[ \tilde{\mathcal{H}}'(r) \right]^2 \right\}^{1/2}
\]

is not a space of functions; rather it is a space of functions modulo the constant functions, since there exist Cauchy sequences in \( C_\infty_c(\mathbb{R}_+) \) that converge to zero in norm but converge pointwise to a non-zero constant function. Therefore it is necessary that \( |\tilde{R}|_{\mathcal{H}} \) control the size of \( \tilde{R} \) as well as its distributional derivative \( \tilde{R}' \). In order that \( F \) be coercive it must also control the size of \( \tilde{R} = R - \tilde{R} \), and the only way that this is possible is through the potential term. Through our requirement on \( V \), the potential term dominates \( \int_0^\infty r dr [\tilde{R}(r)]^2 \) whenever \( 0 \leq \tilde{R} + \tilde{R}_j \leq 1 \), and we can arrange that \( 0 \leq \tilde{R} + \tilde{R}_j \leq 1 \) for the \( \tilde{R}_j \) in a minimizing sequence.

For the Yang-Mills Higgs model, though, \( |\tilde{R}|_{\mathcal{H}} = \left\{ \int_0^\infty r^2 dr [\tilde{R}'(r)]^2 \right\}^{1/2} \) does control \( R(r) \) for large \( r \) (because the measure is \( r^2 dr \) instead of \( rdr \)), so the above requirement on \( V \) is superfluous. On the other hand the finiteness of \( F(R,S) \) does not force \( R(r) \) to vanish as \( r \to 0 \); in fact it is only necessary that \( R(r) = o(r^{-b_2}) \) as \( r \to 0 \) for \( F(R,S) \) to be finite. This causes no difficulty in proving that minima exist, but at first sight it creates havoc in the proof that minima are weak solutions: the term \( t \int_0^\infty r^2 dr \, V'(\tilde{R}(r)) R(r) \), which is added to and subtracted from \( F(R + t\tilde{R}, S) \) to facilitate the calculation of \( \frac{d}{dt} \bigg|_{t=0} F(R + t\tilde{R}, S) \), is not clearly finite. Thus it is crucial to note that \( 0 \leq R \leq 1 \), if \( (R,S) \) minimizes \( F \), in order to prove that minima are weak solutions. (This is what is not mentioned in [TFS], [R1], or [W].) Of course we do not need to note that \( 0 \leq R \leq 1 \) in order to see that \( (R,S) \) is a weak solution in
the less restrictive sense that the equations in Definition III.2.5 hold only for \( R \) and \( S \) with compact support, but it is more convenient for our proof of regularity that we know that \((R,S)\) is a weak solution in the stronger sense of Definition III.2.5.
III.3 The Yang-Mills Higgs Model — Regularity of Solutions

In this section we present a proof that a weak solution of (III.2-1)
gives rise to Higgs and gauge fields \( \phi \) and \( A \) that satisfy (cf. VII.1)
\[
\left(1 + \frac{1}{|x|}\right) \frac{\Phi}{|x|} = 0
\]
and
\[
\frac{d^*}{dx} F(A) + \int_{A} (\Phi) = 0
\]
where \( \phi \) and \( A \) are defined (on \( \mathbb{R}^3 \setminus \{0\} \)) by
\[
\phi(x) = R(|x|) \sum_{\alpha} \tau^\alpha \gamma^\alpha \left( \frac{x}{|x|} \right)
\]
and
\[
A(x) = S(|x|) \sum_{\alpha, \beta, \gamma} \frac{\sigma^\alpha_\beta}{2i} \epsilon_\alpha_\beta_\gamma \frac{x^\gamma}{|x|^2}
\]

Our method of proof is the same as for the Abelian Higgs model, differing
only in some minor details. A different proof was given by Rawnsley [R2]
(for the case \( \ell = 1 \)); we include our proof for the sake of completeness
of our exposition.

Define the truncated Green functions \( G^r_R \) and \( G^r_S \) on \( \mathbb{R}^+ \times \mathbb{R}^+ \) by
\[
G^r_R (r_o, r) = \begin{cases} 
\frac{i}{2 \ell+1} r_o^\ell \frac{1}{r^\ell+1} & \text{if } r \leq r_o \\
\frac{1}{2 \ell+1} r_o^\ell \frac{1}{r^\ell} \chi_\ell (r) & \text{if } r > r_o
\end{cases}
\]
and
\[ G^r_{S'}(r_o, r) = \begin{cases} \frac{1}{3} \frac{r^2}{r_o} & r \leq r_o \\ \frac{1}{3} \frac{r^2}{r} \chi_{r_1}(r) & r > r_o \end{cases} \]

where \( 0 < r_0 < r_1 \) and \( \chi_{r_1} \) is a \( C^\infty \) function on \( \mathbb{R}_+ \) such that \( \chi_{r_1}(r) = 1 \) for \( r \leq r_1 \) and \( \chi_{r_1}(r) = 0 \) for \( r > r_1 + 1 \). Note that \( G^r_{R'}(r_o, \cdot) \in H_R \) and \( G^r_{S'}(r_o, \cdot) \in H_S \).

**Lemma III.3.1.** If \((R, S)\) is a weak solution of (III.7-1), then there are constants \( \alpha \) and \( \beta \) such that

\[
R(r_o) = \int_0^{r_o} r^2 dr \left\{ \frac{1}{2} \frac{r^2}{r^2} R(r) (S(r) + 1) R(r) + \frac{r}{r^2} (V(r) + V'(r, r)) \right\} G^r_{R'}(r_o, r) + \alpha r_o^2 = 0
\]

and

\[
S(r_o) = \int_0^{r_o} r^2 dr \left\{ \frac{1}{2} \frac{r^2}{r^2} R(r) (S(r) + 1) + r^{-4} S'(r) (S(r) + 1) \right\} G^r_{S'}(r_o, r) + \beta r_o^2 = 0.
\]

**proof:** Since \( R(r) = O(r^{-2}) \) as \( r \to 0 \),

\[
r_0^{l+1} R(r_o) = \int_0^{r_o} dr \cdot \frac{d}{dr} \left[ R(r) r^L \right].
\]

Therefore

\[
\int_0^{r_o} r^2 dr \frac{d}{dr} \left( \frac{G^r_{R'}(r_o, r)}{r_o} \right) = \frac{1}{2L+1} \int_0^{r_o} dr \ R'(r) \left( \frac{r}{r_o} \right)^{L+1} - \frac{1}{2L+1} \int_0^{r_o} dr \ R''(r) \left( \frac{r}{r_o} \right)^{L+1} \chi_{r_1}(r)
\]

\[
+ \frac{1}{2L+1} \int_0^{r_o} dr \frac{r^L}{2L+1} R'(r) \chi_{r_1}'(r)
\]

\[
= \frac{1}{2L+1} \int_0^{r_o} dr \cdot \frac{d}{dr} \left[ R(r) \left( \frac{r}{r_o} \right)^{L+1} \right] - \frac{1}{2L+1} \int_0^{r_o} dr \ L(L+1) R(r) \cdot \frac{r^L}{r_o} \chi_{r_1}'(r)
\]

\[
- \frac{L+1}{2L+1} \int_0^{r_o} dr \cdot \frac{d}{dr} \left[ R(r) \left( \frac{r}{r_o} \right)^{L+1} \chi_{r_1}'(r) \right] - \frac{1}{2L+1} \int_0^{r_o} dr \ L(L+1) R(r) \cdot \frac{r^L}{r_o} \chi_{r_1}'(r)
\]
\[ + r_0 ^k \int_0 ^r \frac{dv}{v^{k+1}} r^{-\frac{k+1}{2}} R(v) \chi'_{r_1} (r) + r_0 ^k \int_0 ^r \frac{r - \frac{k+1}{2} R'(r) \chi'_{r_1} (r)}{v^{k+1}} \]

\[ = R(r_0) - \int_0 ^r v^2 dv A(l,h) r^{-2} R(r) G_{R}^{r_1} (r_0, r) + \alpha r_0 ^k , \]

where \( \alpha \) is a constant independent of \( r_0 \). Similarly

\[ \int_0 ^r v^2 dv \cdot S'(r) \frac{\partial G_{S}^{r_1}}{\partial r} (r_0, r) = \frac{2}{3} \int_0 ^r dv S'(r) \frac{r}{v} - \frac{1}{3} \int_0 ^r dv S'(r) \left( \frac{r}{v} \right)^2 \chi'_{r_1} (r) \]

\[ + r_0 ^k \int_0 ^r dv S'(r) \frac{1}{3} r^{-1} \chi'_{r_1} (r) \]

\[ = \frac{2}{3} \int_0 ^r dv \left[ S(r) \frac{r}{v} \right] - \int_0 ^r dv 2v^{-2} S(r) \cdot \frac{r}{3} \frac{r^2}{v} \]

\[ - \frac{1}{3} \int_0 ^r dv \left[ S(r) \left( \frac{r}{v} \right)^2 \chi'_{r_1} (r) \right] - \int_0 ^r dv 2v^{-2} S(r) \cdot \frac{1}{3} \frac{r^2}{v} \chi'_{r_1} (r) \]

\[ + r_0 ^k \int_0 ^r dv S(r) \frac{1}{3} r^{-1} \chi'_{r_1} (r) + r_0 ^k \int_0 ^r dv S'(r) \frac{1}{3} r^{-1} \chi'_{r_1} (r) \]

\[ = S(r_0) - \int_0 ^r v^2 dv \cdot 2v^{-4} S(r) G_{S}^{r_1} (r_0, r) + \beta r_0 ^k . \]

Because \((R,S)_r\) is a weak solution of (III.2-1) and because \(G_{R}^{r_1} (r_0, r)\) and \(G_{S}^{r_1} (r_0, r)\) satisfy the requirements of \(R\) and \(S\) in Definition III.2.5

we find that

\[ \int_0 ^r v^2 dv R'(r) \frac{\partial G_{R}^{r_1}}{\partial r} (r_0, r) + \int_0 ^r dv \left[ A(l,h) r^{-2} (S(r)+1)^2 R(r) + V'(R(r)) \right] G_{R}^{r_1} (r_0, r) = 0 \]

and

\[ \int_0 ^r v^2 dv S'(r) \frac{\partial G_{S}^{r_1}}{\partial r} (r_0, r) + \int_0 ^r dv \left[ \frac{1}{2} A(l,h) r^{-2} R'(r) (S(r)+1) + r^{-4} S(r) (S(r)+1)^2 \right] G_{S}^{r_1} (r_0, r) \]

\[ = 0 . \]
Comparing these formulae with the ones just proved yields (III.3-3).

Theorem IV.3.2. Suppose $(R, S)$ is a weak solution of (III.2-1). Then there exist $C^2$ fields $\phi$ and $A$ on $\mathbb{R}^3$ that extend the definition (III.3-2) and satisfy (III.3-1).

Proof: Write (III.3-3) in the form

$$R(r_o) + \int_{r_o}^{R} r^2 dr \left\{ \frac{1}{2} \lambda (l+1) r^2 \left( S(r) (S(r) + 2) R(r) + V'(R(r)) \right) \right\} \frac{r^l}{2^{l+1}} \frac{1}{r^l} \frac{r^l}{r^l}$$

$$+ \int_{r_o}^{r} r^2 dr \left\{ \frac{1}{2} \lambda (l+1) r^2 \left( S(r) (S(r) + 2) R(r) + V'(R(r)) \right) \right\} \frac{r^l}{2^{l+1}} \frac{1}{r^l} \frac{r^l}{r^l}$$

$$+ \alpha' r^l_r = 0$$

and

$$S(r_o) + \int_{r_o}^{R} r^2 dr \left\{ \frac{1}{2} \lambda (l+1) r^2 \left( S(r) (S(r) + 1) + r^{-4} S^2(r) (S(r) + 3) \right) \right\} \frac{r^l}{3} \frac{1}{r^l} \frac{r^l}{r^l}$$

$$+ \int_{r_o}^{r} r^2 dr \left\{ \frac{1}{2} \lambda (l+1) r^2 \left( S(r) (S(r) + 1) + r^{-4} S^2(r) (S(r) + 3) \right) \right\} \frac{r^l}{3} \frac{1}{r^l} \frac{r^l}{r^l}$$

$$+ \beta' r^l_r = 0,$$

where $0 < r_0 < r_1$ and the constants $\alpha'$ and $\beta'$ are independent of $r_0$.

Since the integrands in these expressions are continuous on $\mathbb{R}^+$, we may calculate the derivatives of $R$ and $S$ to be

$$R'(r_o) = (l+1) r_o^{-1} \int_{r_o}^{r} r^2 dr \left\{ \frac{1}{2} \lambda (l+1) r^2 S(r) (S(r) + 2) R(r) + V'(R(r)) \right\} \frac{r^l}{2^{l+1}} \frac{1}{r^l} \frac{r^l}{r^l}$$

$$- \lambda r_o^{-l} \int_{r_o}^{r} r^2 dr \left\{ \frac{1}{2} \lambda (l+1) r^2 S(r) (S(r) + 2) R(r) + V'(R(r)) \right\} \frac{r^l}{2^{l+1}} \frac{1}{r^l} \frac{r^l}{r^l}$$

$$- \lambda \alpha' r_o^{l-1}$$
and

\[ S'(v_o) = \int_{0}^{v_o} r^2 \, dv \left\{ \frac{1}{2} R^2(r) \left( S(r+1) \right) + R^2(r) \left( S(r+3) \right) \right\} \cdot \frac{1}{3} \frac{r^2}{v_o^2} \]

\[ - 2 \int_{0}^{v_o} r^2 \, dv \left\{ \frac{1}{2} L (l+1) R^2(r) \left( S(r+1) \right) + R^2(r) \left( S(r+3) \right) \right\} \cdot \frac{1}{3} \frac{r^2}{v_o} \]

\[ - 2 \beta' R_o \]

thus \( R' \) and \( S' \) are continuous on \( \mathbb{R}_+ \), and we may differentiate again to find that \( R \) and \( S \) satisfy (III.2-1) and are \( C^2 \) on \( \mathbb{R}_+ \). Therefore \( \phi \) and \( A \), as defined by (III.3-2), are \( C^2 \) on \( \mathbb{R}^3 \setminus \{0\} \) and satisfy (III.3-1) there.

Note that the components of \( A \) are

\[ A_{a,k} (x) = \sum_{j} \varepsilon_{a,j} \kappa \cdot x^{j} \cdot |x|^{-1} S\left( |x| \right) \]

Thus to prove that \( A \) extends to a \( C^2 \) field on \( \mathbb{R}^3 \), it suffices to verify that the function \( r \mapsto r^{-\ell} S(r) \) satisfies the properties required of \( f \) in Lemma II.4.3. Similarly it suffices to show that \( r \mapsto r^{-\ell} R(r) \) satisfies these properties since

\[ \phi_m (x) = |x|^{\ell} Y_m^l \left( \frac{x}{|x|} \right) \cdot |x|^{-\ell} R\left( |x| \right) \]

and since \( |x|^{\ell} Y_m^l \left( \frac{x}{|x|} \right) \) is a polynomial in \( x_1, x_2, \) and \( x_3 \) which is homogeneous of degree \( \ell \geq 1 \). For this purpose we rewrite (III.3-3) as

\[ r_o^{-\frac{\ell}{2}} R(r_o) + \int_{0}^{v_o} dr \left\{ \frac{1}{2} R^2(r) \left( S(r+1) \right) + \frac{V' \left( R(r) \right)}{R(r)} \right\} \cdot \frac{1}{2k+1} \left( \frac{r}{r_o} \right)^{\frac{\ell+1}{2}} \cdot r^{-\frac{k}{2}+1} R(r) \]

\[ + \int_{0}^{v_o} dr \left\{ \frac{1}{2} L (l+1) R^2(r) \left( S(r+1) \right) + \frac{V' \left( \beta(r) \right)}{R(r)} \right\} \cdot \frac{1}{2k+1} \left( \frac{r}{r_o} \right)^{\frac{\ell-1}{2}} \cdot r^{-\frac{k+1}{2}+1} R(r) \]

\[ + \alpha' R_o^{-\frac{\ell}{2}} = 0 \]
\[ r^k \left( \sum_{i=0}^{j-1} r_i \right) + \int_{r_0}^{r} r^2 \, dr \left\{ \frac{j}{2} \frac{r^2}{(r+1)^2} R(r) (S(r)+1) + r^{-2} S^2(r) (S(r)+3) \right\} \frac{r^3}{(r_0^3)} r^k \frac{r^2}{(r_0^2)} \]

\[ + \int_{r_0}^{r} r^2 \, dr \left\{ \frac{j}{2} \frac{r^2}{(r+1)^2} R(r) (S(r)+1) + r^{-2} S^2(r) (S(r)+3) \right\} \frac{r^3}{(r_0^3)} r^k \frac{r^2}{(r_0^2)} \]

\[ + \beta r_0 \frac{k}{2} + 2 = 0, \]

where \( j \) and \( k \) are non-negative integers. Since \( S(r) \to 0 \) as \( r \to 0 \) the finiteness of \( F(R,S) \) shows that we may let \( r_0 \to 0 \) in the second formula when \( k = 2 \), so that \( S(r) = O(r) \) as \( r \to 0 \). From this we conclude that

\[ r \mapsto \left\{ 2r^{-2} S(r) (S(r)+2) + \frac{V'(R(r))}{R(r)} \right\} r^k \]

is integrable at zero, since \( V'(R_1) = V''(0) R_1 + o(R_1) \) as \( R_1 \to 0 \); thus the first formula, with \( j = 1 \), shows that \( R(r) = O(r^{1/2}) \) as \( r \to 0 \).

Putting this result into the second formula with \( k = 3 \) shows that \( S(r) = O(r^{3/2}) \) as \( r \to 0 \), and from this and the first formula with \( j = 2 \) we find that \( \lim_{r \to 0} r^{-1} R(r) \) exists. Finally the second formula with \( k = 4 \) yields that \( \lim_{r \to 0} r^{-2} S(r) \) exists, and induction on \( j \) shows that \( \lim_{r \to 0} r^{-k} R(r) \) exists. Moreover,

\[ \frac{d}{dr_0} \left[ r_0^{-1} R(r_0) \right] = \int_{r_0}^{r} \, dr \left\{ \ell (l+1) r^{-2} S(r) (S(r)+2) + \frac{V'(R(r))}{R(r)} \right\} \left( \frac{r}{r_0} \right)^{2(l+1)} r^k \frac{r^2}{r_0^2} \]

\[ = r_0 \left\{ \ell (l+1) r_0^{-2} S(r_0) (S(r_0)+2) + \frac{V'(R(r_0))}{R(r_0)} \right\} r_0^{-k} R(r_0) \]

\[ - 2 \ell (l+1) \frac{1}{dr_0} \left[ r_0^{-1} R(r_0) \right] \]
\[
\frac{d}{dr_0} \left[ r_o^{-2} S(r_o) \right] = \int_0^{r_0} \frac{1}{2} \left\{ \frac{1}{2} \lambda (\lambda + 1) r_s^{-2} R'(r) (S(r) + 1) + r_s^{-4} S(r) (S(r) + 3) \right\} \left( \frac{r_s}{r_o} \right)^4 dr,
\]

and

\[
r_o \frac{d^2}{dr_o^2} \left[ r_o^{-2} S(r_o) \right] = r_o \left\{ \frac{1}{2} \lambda (\lambda + 1) r_s^{-2} R'(r_o) (S(r_o) + 1) + r_o^{-4} S(r_o) (S(r_o) + 3) \right\}
- 4 \frac{d}{dr_o} \left[ r_o^{-2} S(r_o) \right],
\]

so that the limits as \( r_o \to 0 \) of these quantities all vanish. Therefore the fields \( \phi \) and \( A \) have \( C^2 \) extensions to \( \mathbb{R}^3 \), also denoted \( \phi \) and \( A \), and these extensions satisfy (III.3-1) on \( \mathbb{R}^3 \) by continuity. \( \square \)
11.4 The Yang-Mills Higgs Model - Behavior of Solutions

In the following we will apply the results of 11.5 to determine the asymptotic behavior at infinity of the isotropic solutions of the Yang-Mills Higgs equations that we have constructed. We find that \( R(r) \to 1 \) and \( S(r) \to -1 \) as \( r \to \infty \), so that the Higgs phenomenon takes place and the monopole number is

\[
\kappa = 2 \lim_{r \to \infty} R(r) S(r) \left( \frac{1}{2} S(r) + 1 \right) = -1.
\]

Moreover \( R(r) \) and \( S(r) \) approach their limits exponentially fast: the inverse characteristic length of the exponential approach for \( S \) is \( \frac{1}{2} \zeta (\ell + 1) \), whereas for \( R \) it is \( \min \{ \zeta''(1) \zeta, 2 \zeta (\ell + 1) \} \). Thus the classical Yang-Mills Higgs model reflects the decay mode of the quantum model in which a Higgs particle decays into two massive gauge particles, just as does the Abelian Higgs model.

It is convenient to introduce the shifted fields \( u \) and \( v \) defined by

\[
\begin{align*}
    U(r) &= r (1 - R(r)) \\
    V(r) &= 1 + S(r).
\end{align*}
\]

Given that \( R \) and \( S \) satisfy (11.2-1), \( u \) and \( v \) satisfy

\[
-u''(r) + \left[ \frac{1}{2} \zeta (\ell + 1) - r^{-2} - \zeta (\ell + 1) r^{-1} U(r) + \frac{1}{2} \zeta (\ell + 1) V^{-2} U(r) + r^{-2} V^2(r) \right] V(r) = 0,
\]

and

\[
-v''(r) + \left[ \frac{1}{2} \zeta (\ell + 1) - r^{-2} - \zeta (\ell + 1) r^{-1} U(r) + \frac{1}{2} \zeta (\ell + 1) V^{-2} U(r) + r^{-2} V^2(r) \right] U(r) = 0,
\]

where we have written \( V'(1 - R) = -V''(1) R + \hat{V}(\hat{R}) \hat{R}^2 \). By Taylor's theorem
\( \tilde{v}(\tilde{R}) = O(1) \) as \( \tilde{R} \to 0. \)

**Theorem III.4.1.** Suppose \((R, S) \in \mathbb{N} \) is a \( C^2 \) solution of (III.2-1) such that \( F(R, S) \) is finite; let \( m = [V''(1)]^{1/2}, m_\perp = [\frac{2}{3}(\ell + 1)]^{1/2}, \) and \( \nu_0 = i \frac{3^2}{2} / 2. \) Then there exist constants \( \alpha \) and \( \beta \) such that

\[
S(r) = -1 + \beta (m_r) \frac{1}{2} K_{\nu_0} (m_r) \cdot \ln \left( 1 + o \left( \exp \left( -\min \{ m, 2m_\perp^2 \} r \right) \right) \right)
\]

as \( r \to \infty, \) and such that:

(a) \( \text{if } m < 2m_\perp, \)

\[
R(r) = 1 - \frac{\alpha m r}{r^2} - \frac{m r}{r} \left( 1 + o \left( \exp \left( -2m r \right) \right) \right) - 2m_\perp \left( m_\perp^2 + 4m_\perp^2 \right) K_{\nu_0}^2 (m_r) \left( 1 + o(1) \right);
\]

(b) \( \text{if } m = 2m_\perp, \)

\[
R(r) = 1 - \frac{1}{2} m_\perp \beta \left( m_r \right) \ln \left( K_{\nu_0} (m_r) \right)^2 \left( 1 + o(1) \right);
\]

and

(c) \( \text{if } m > 2m_\perp, \)

\[
R(r) = 1 - 2m_\perp \left( m_\perp^2 + 4m_\perp^2 \right) K_{\nu_0}^2 (m_r) \left( 1 + o(1) \right)
\]

as \( r \to \infty. \)

**proof:** Define the fields \( u \) and \( v \) in terms of \( R \) and \( S \) by (III.4-1); then \( u \) and \( v \) satisfy (III.4-2). By hypothesis \( R - R_{\ell} H_{R} \), so Lemma III.2.1 shows that \( R(r) = 1 + 0(r^{-2}) \) as \( r \to \infty. \) Since \( V(R_1) = V''(1)(R_1-1)^2 + o(R_1-1)^2 \) as \( R_1 \to 1 \) and \( V''(1) > 0 \) there is a constant \( \alpha > 0 \) such that

\[
V(R_1) \geq \alpha \left( 1 - R_1 \right)^2
\]

for all \( R_1 \) near 1. Thus we see that
\[
\int_{r_0}^{\infty} r^2 \, dr \, V(R(r)) \geq \alpha \int_{r_0}^{\infty} r^2 \, dr \, (1 - R(r))^2 = \alpha \int_{r_0}^{\infty} \dot{r} \, u^2(r)
\]
for large enough \( r_0 \), so that the finiteness of \( F(R,S) \) guarantees that \( u \) is square-integrable at infinity. We also conclude that \( u' \), given by

\[
u'(r) = -r \left( R - \dot{R} \right)'(r) + r^{-1} \nu(r)
\]
for \( r \geq 2 \), is square-integrable at infinity since \( R - \dot{R} \in H_R \). Therefore \( \nu^2 = 2u \cdot u' \) is integrable at infinity, whence \( \lim_{r \to \infty} u(r) \) exists and thus vanishes. Since the third term

\[
\int_{r_0}^{\infty} r^3 \, dr \cdot \frac{1}{2} \lambda(l+1) r^{-2} \left( S(r) + \dot{S} \right)^2 R^2(r) = \int_{r_0}^{\infty} \, dr \cdot \frac{1}{2} \lambda(l+1) \nu^2(r) \left( 1 - r^{-1} \nu(r) \right)
\]
in \( F(R,S) \) is finite it follows that \( \nu \) is square-integrable at infinity.

By assumption again \( S - \hat{S} \in H_S \), so that \( \nu' \), given by

\[
u'(r) = (S - \hat{S})'(r)
\]
for \( r \geq 2 \), is square integrable at infinity. As before we conclude that \( \nu(r) \to 0 \) as \( r \to \infty \).

Examining (III.4-2) shows that Lemma II.5.1 is applicable to \( \nu \): given any positive \( \epsilon < 1 \)

\[
u(r) = O \left( \exp \left[ - \frac{m_{\lambda}}{2} (1 - \epsilon)^{1/2} r \right] \right)
\]
as \( r \to \infty \). From this bound and Lemma II.5.1 applied to \( u \) we see that

\[
u(r) = O \left( \exp \left[ - \min \{ m_{\lambda}, m_{\mu} \} (1 - \epsilon)^{1/2} r \right] \right)
\]
as $r \to \infty$.

Again referring to (III.4-2) we find that Lemma II.5.2 applies to the equation satisfied by $v$. Since $v$ vanishes at infinity it must be proportional to the subdominant solution, i.e. there exists a constant $\beta'$ such that

$$v(r) = \beta' \frac{1}{\nu_0} \left( m \frac{r}{l} \right)^{1/2} \left( 1 + o \left( \exp \left[ -\min \left\{ m, 2m_x \right\} (1 - \varepsilon)^{1/2} r \right] \right) \right)$$

as $r \to \infty$.

Consider the equation satisfied by $u$, written in the form

$$- u''(r) + \left[ m^2 + b_2(r) \right] u(r) = g(r).$$

Let $u_+$ and $u_-$ denote the solutions of the corresponding homogeneous equation that are constructed in Lemma II.5.2. Let

$$u_p(r) = (2m)^{-1} \int_r^\infty \left[ \int_r^\infty \frac{u_+(r') u_-(r) g(r') \, dr'}{r} \right] \, dr$$

for $r \geq r_1$; as shown in the proof of Theorem II.5.3, $u_p$ is a particular solution of the equation satisfied by $u$. Thus there are constants $\alpha_+$ and $\alpha_-$ such that $u = \alpha_+ u_+ + \alpha_- u_- + u_p$. We will determine the asymptotic behavior of $u_p$ just as in the proof of Theorem II.5.3.

By l'Hôpital's rule,

$$\lim_{r \to \infty} \frac{(2m)^{-1} \int_r^\infty \frac{u_+(r') g(r') \, dr'}{g(r)} \, dr}{ \frac{1}{q(r')} \left[ u_+(r) \right]^{-1} \frac{q(r')}{g(r)}} = (2m)^{-1} \left[ m - \lim_{r \to \infty} \frac{q(r')}{g(r)} \right]^{-1};$$

but with $g(r) = k(k + 1) r^{-1} v^2(r)$,
\[ \frac{q'(r)}{q(r)} = -r^{-1} + 2v'(r)/v(r) = 2m \, \delta - r^{-1} + O(r^{-2}) \]

as \( r \to \infty \) (by Lemma 11.5.2). Thus the second term in \( u_\beta \) exhibits the behavior

\[ (2m)^{-1} \int_r^\infty dv' \, u_\beta (v') \, q(v') = (2m)^{-1} (m + 2m_\delta)^{-1} q(r) \cdot (1 + o(1)) \]

as \( r \to \infty \).

Suppose \( m < 2m_\delta \). Then the first term in \( u_\beta \) may be written

\[ (2m)^{-1} \int_r^\infty dv' \, u_\beta (v') \, q(v') \, \cdot u_\beta (r) = (2m)^{-1} \int_r^\infty dv' \, u_\beta (v') \, q(v') \]

because \( u_\beta \cdot q \) vanishes exponentially at infinity. But

\[ \lim_{r \to \infty} \frac{- (2m)^{-1} \int_r^\infty dv' \, u_\beta (v') \, \frac{q(v')}{q(r)} [u_\beta (r)]^{-1}}{q(r)} = (2m)^{-1} \left[ m + \lim_{r \to \infty} \frac{q'(r)}{q(r)} \right]^{-1} \]

\[ = (2m)^{-1} (m - 2m_\delta)^{-1}, \]

so that there is a constant \( \alpha'' \) such that \( u_\beta (r) = \alpha' u_\beta (r) + [m^2 - 4m_\delta^2]^{-1} q(r) \cdot (1 + o(1)) \)

as \( r \to \infty \). Since \( u \) vanishes at infinity we conclude that with \( \alpha' = \alpha' + \alpha'' \),

\[ u(r) = \alpha' \frac{L}{\sqrt{\lambda}}(m \, r) \cdot \left( 1 + o \left( \exp \left[ - m \left( 1 - \varepsilon \right)^{1/2} \, r \right] \right) \right) \]

\[ + \left[ m^2 - 4m_\delta^2 \right]^{-1} \cdot \ell(\ell + 1) \cdot r^{-1} \left[ \beta' \frac{L}{\sqrt{\lambda}}(m \, r) \right]^2 \cdot (1 + o(1)) \]

as \( r \to \infty \). If on the other hand \( m > 2m_\delta \), then

\[ \lim_{r \to \infty} \frac{(2m)^{-1} \int_r^r dv' \, u_\beta (v') \, q(v')}{q(r) \, [u_\beta (r)]^{-1}} = (2m)^{-1} \left[ m + \lim_{r \to \infty} \frac{q'(r)}{q(r)} \right] \]

\[ = (m)^{-1} (m - 2m_\delta)^{-1}. \]
since \( g \cdot [u_-]^{-1} \) grows (exponentially) at infinity; thus

\[
u(r) = \alpha_- k_{\frac{m}{2}} (mr) \cdot \left( 1 + o \left( \exp \left[ -2 \frac{m}{2} (1 - \varepsilon)^{\frac{1}{2}} r \right] \right) \right)
\]

\[
+ \left[ m^2 \left[ m_r \right]^{-1} k_{\frac{m}{2}} (m_r) \right] \cdot \left( \frac{\beta'}{\nu_0} (m_r) \right)^2 \cdot (1 + o(1))
\]

as \( r \to \infty \). Lastly, in case \( m = 2m_\varepsilon \), \( r \ln(m_r) g(r) [u_- (r)]^{-1} \) grows (logarithmically) as \( r \to \infty \), so that

\[
\lim_{r \to \infty} \left( \frac{2m}{r} \right) \int_{r_1}^{r} \frac{\dot{u}_+ (r') g(r')}{r \ln(m_r) g(r) \left[ u_- (r) \right]^{-1}} = \lim_{r \to \infty} \frac{(2m)^{-1} u_+ (r) g(r)}{r \ln(m_r) g(r) \left[ u_- (r) \right]^{-1}} \left( \frac{r^{-1} \ln(m_r) g(r) \left[ u_- (r) \right]^{-1} + \frac{\dot{u}_+ (r)}{g(r)} - \frac{u_- (r)}{g(r)}}{r \ln(m_r) g(r) \left[ u_- (r) \right]^{-1}} \right)
\]

\[
= (2m)^{-1} \lim_{r \to \infty} \left[ r \ln^2(m_r) \left\{ \left[ r \ln(m_r) \right]^{-1} + O(r^{-1}) \right\} \right]^{-1} = (2m)^{-1}
\]

because \( g'(r)/g(r) = -2m_\varepsilon - r^{-1} + O(r^{-2}) \) as \( r \to \infty \); therefore

\[
\nu(r) = \alpha_- k_{\frac{m}{2}} (mr) \cdot \left( 1 + o \left( \exp \left[ -m (1 - \varepsilon)^{\frac{1}{2}} r \right] \right) \right)
\]

\[
+ (2m)^{-1} r \ln(m_r) \cdot k_{\frac{m}{2}} (m_r) \cdot \left[ \beta' \nu_0^2 (m_r) \right] \cdot (1 + o(1))
\]

as \( r \to \infty \).

From these formulae and Lemma 11.5.2 we see that we may replace the \( (1 - \varepsilon)^{\frac{1}{2}} \) by 1. Finally, using the definitions of \( u \) and \( v \) in terms of \( R \) and \( S \), and noting that \( k_{\frac{m}{2}} (mr) = \exp(-mr) \) and \( \nu_0 (m_r) = (2m_\varepsilon r/\pi)^{\frac{1}{2}} k_{\frac{m}{2}} (m_r) \) finishes the proof. []

Clearly \( \beta \) is non-zero because \( v \) does not vanish identically; but we cannot determine the sign of \( \beta \) as we did for the Abelian Higgs model.
because \( v \) is not necessarily monotone on all of \( \mathbb{R}_+ \). Of course \( \beta \)
is positive if \( S(r) \geq -1 \) for all large \( r \). It is, however, possible to
prove that \( |v| \leq 1 \), i.e. that \(-2 \leq S \leq 0\), by using the maximum principle
argument of Lemma II.5.1. (This result is proved in a different way in \([W]\).)
Indeed if \( v^2(r_1) > 1 \) at some \( r_1 \in \mathbb{R}_+ \) then either \( v \) attains a maximum
\( v(r_{\text{max}}) > 1 \) or a minimum \( v(r_{\text{min}}) < -1 \) (since \( v(r) \to 1 \) as \( r \to 0 \) and
\( v(r) \to 0 \) as \( r \to \infty \)); but this contradicts the equation

\[
v''(r) = \left[ \frac{1}{2} \lambda l(l+1) R^2(r) + r^{-2} \left( \frac{v(r)}{r} - 1 \right) \right] v(r),
\]

satisfied by \( v \).

As for the constant \( \alpha \), it must be positive assuming either that
\( R(r) \leq 1 \) for large \( r \) or that \( R \geq 0 \) and

\[
v'(R_1) + v''(1)(1 - R_1) \geq 0
\]

for all \( R_1 \geq 0 \). This latter result is proved in the same way as in the
Abelian Higgs case:

\[
-u''(r) + m^2 u(r) = r \left[ \lambda l(l+1) r^{-1}(S(r)+1)^2 R(r) + V'(R(r)) + V''(1)(1-R(r)) \right]
\]

so that

\[
\left[ m \cosh (mr) \ u(r) - \sinh (mr) \ u'(r) \right]_0^\infty \int_0^\infty dr \ \sinh (mr) \ f_u(r);
\]

but if \( m < 2m_0 \), \( u(r) = \alpha e^{\exp(-mr)} \cdot (1 + o(1)) \) and \( u'(r) = -m \alpha e^{\exp(-mr)} \cdot (1+o(1)) \)
as \( r \to \infty \), while \( u(r) = r(1 + o(1)) \) and \( u'(r) = 1 + o(1) \) as \( r \to 0 \).
so that we find
\[ \alpha = m a' = \int_0^\infty dr \sinh(mr) f_\nu(r), \]

whence \( \alpha \) is positive under the conditions above.
IV. Conclusion

We summarize what we have proved in the following two theorems.

Theorem IV.1. Suppose \( V \) is a \( C^2 \), nonnegative, symmetric function on \( \mathbb{R} \) such that \( V(1) = 0 \) and \( m = [V''(1)]^{1/2} > 0 \); suppose further that \( V \) does not vanish on \( [0,1] \). Then for each integer \( n \) there exist finite action, isotropic solutions of the Abelian Higgs equations with vortex number \( n \). More precisely, there exist Higgs and gauge fields \( \phi \) and \( A \) such that:

1. \( \phi \) is a \( C^2 \), complex-valued function on \( \mathbb{R}^2 \) and \( A \) is a \( C^2 \), real-valued 1-form on \( \mathbb{R}^2 \) of the form
   \[
   \phi(x, y) = R(r) e^{i n \theta}
   \]
   and
   \[
   A(x, y) = S(r) \, d\theta
   \]
   which satisfy (see II.1 for the notation)
   \[
   \partial^* A \phi + \frac{1}{2} \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi} - \frac{1}{2} \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi} = 0;
   \]
   and
   \[
\int d^2 \alpha \cdot A + \frac{1}{2} \kappa \left( \overline{\phi} \frac{\partial}{\partial A} \phi - \phi \frac{\partial}{\partial A} \overline{\phi} \right) = 0;
\]

2. the action
   \[
   \mathcal{F}(\phi, A) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \| A \|^2 + \frac{i}{2} \| \phi \|^2 + \| A \| \right\};
   \]
   is finite;

3. the vortex number
   \[
   \frac{i}{2\pi} \int_{\mathbb{R}^2} \tilde{d}(\xi A)
   \]
is the integer $n$;

and (4) the functions $R$ and $S$ have the properties that

(i) $0 \leq R \leq 1$, $-\mid n \mid \leq \text{sgn } n \cdot S \leq 0$, and $\text{sgn } n \cdot S$ is monotonically decreasing,

and (ii) there exist constants $\alpha > 0$ and $\beta > n$ such that as $r \to \infty$,

$$S(r) = -n + \text{sgn } n \cdot \beta r K_1(r) \cdot \left(1 + o\left(\exp \left[-\min \{m, 2\} r\right]\right)\right)$$

and

$$R(r) = 1 - \left\{\frac{1}{2} \beta^2 r \left[K_1(r)\right]^2 \cdot (1 + o(1))\right\}$$

$$- \left[\frac{m^2 - 4}{\beta^2} \left[K_1(r)\right]^2 \cdot (1 + o(1))\right]$$

according as $m < 2$, $= 2$, $> 2$.

Theorem IV.2. Suppose $V$ is a $C^2$, non-negative, symmetric function on $\mathbb{R}$ such that $V(1) = 0$ and $m = [V''(1)]^{1/2} > 0$. Then for each integer $\ell$ there exist finite action, isotropic solutions of the Yang-Mills Higgs equations with isospin $\ell$. More precisely, there exist Higgs and gauge fields $\phi$ and $A$ such that:

(1) $\phi$ is a $C^2$ function on $\mathbb{R}^3$ taking values in a $(2\ell+1)$-dimensional vector space $W$, and $A$ is a $C^2$ 1-form on $\mathbb{R}^3$ taking values in the Lie algebra $su(2)$ of $SU(2)$, which are of the form

$$\phi(x) = R(\|x\|) \sum_{m'} \tau_{m'}^* Y_{m'}^*(\frac{x}{\|x\|})$$
and

\[ A(x) = \sum_{\substack{j, k \in \mathbb{C}^2 \setminus \{0\} \times \mathbb{Z}\}}^\infty \left( \frac{e^{2i}}{\alpha_j k} \right) \left( x^j \sum_{k} \right) \frac{e^{2i}}{\alpha_j k} \left( x^k \right) \]

where the \( \tau^m, m = -l, \ldots, l, \) form a basis for \( W \) which transforms according to the \( \ell \)th irreducible representation of \( SU(2) \), and the \( \alpha^a, a = 1, 2, 3, \) are the Pauli spin matrices; \( \phi \) and \( A \) satisfy (see III.1 for the notation)

\[ j_A \phi + \nabla'_\| \phi \| \frac{\phi}{\| \phi \|} = 0 \]

and

\[ j^*_A F(A) + j_A (\phi) = 0. \]

(2) the action

\[ \mathcal{F}(\phi, A) = \int_{\mathbb{R}^4} \left\{ \frac{1}{2} \| F(A) \|^2 + \frac{1}{2} \| \phi \|^2 + \lambda' \| \phi \| \right\} \]

is finite;

(3) the monopole number (defined in III.1) is \( -1 \);

and (4) the functions \( R \) and \( S \) have the properties that

(1) \( 0 \leq R \leq 1 \) and \( -2 \leq S \leq 0 \),

and (11) there exist constants \( \alpha > 0 \) and \( \beta > 0 \) such that as \( r \to \infty \),

\[ S(r) = -1 + \beta (\lambda r)^{1/2} K_{\nu_0} (\lambda r) \left( 1 + o \left[ \min \{ \lambda, 2\lambda r \} \right] \right) \]
and
\[
\left[ \omega(m \gamma^0 \gamma^r r - m^r r) \left( 1 + \exp(- m r) \right) \right] + 2 m^4 \left[ \frac{m^2 - 4 m^2 \beta^0 (m \gamma^r r)}{\nu_0^2} \right]^2 \left( 1 + o(1) \right)
\]
\[
R(r) = 1 - \left\{ \frac{1}{2} m^2 \beta^0 \left[ \nu_0^2 \left( m \gamma^r r \right) \right]^2 \left( 1 + o(1) \right) \right\}
\]
\[
2 m^4 \left[ \frac{m^2 - 4 m^2 \beta^0 (m \gamma^r r)}{\nu_0^2} \right]^2 \left( 1 + o(1) \right)
\]

according as \( m < 2m_k = 2m_\phi > 2m_\phi \), where \( m_\phi = \left( \frac{\lambda}{2} \lambda + 1 \right)^{1/2} \) and \( \nu_0 = i^{3/2}/2 \).

Let us comment on the physics behind these results. In the quantum field theory of a free scalar field \( \phi \) with mass \( m \) in \( d \)-dimensional Euclidean space the two-point correlation function exhibits the behavior
\[
\langle \phi(x) \phi(y) \rangle = \alpha |x - y|^{-d/2} K_{d/2} \left( m |x - y| \right)
\]
\[
= \left( \frac{\pi^{d/2}}{2m^d} \right) |x - y|^{-d/2} \exp \left( -m |x - y| \right) \left( 1 + o(1) \right)
\]
as \( |x - y| \to \infty \) for a certain constant \( \alpha \). For example we find the familiar Yukawa potential \( r^{-1} \exp(-mr) \) if \( d = 3 \). For interacting quantum fields the physical mass is given in this fashion by the asymptotic behavior as \( |x - y| \to \infty \) of the two point correlation function, rather than by a parameter appearing in the Lagrangian. In analogy with this we consider the exponential decay of our solutions of the classical field equations to give the classical approximation to the mass of the corresponding quantum field.
In fact we find just this behavior for \( R = |\phi| \) so long as \( m < 2 \) in the Abelian Higgs case and \( m < 2m_A \) is the Yang-Mills Higgs case: the mass of the Higgs field is \( m = \sqrt{V''(1)} \). For the gauge field we must be careful to examine the field strength \( F(A) \), which is physically measurable, rather than \( A \) directly. For example, the magnetic field \( B = *dA \) in the Abelian Higgs model is (with \( z \) as in the proof of Theorem II.5.3)

\[
B(r) = r^{-1} S'(r) = \beta \left( K_1'(r) + r^{-1} K_1(r) z(r) \right) + \beta K_2(r) z'(r)
\]

\[
= \beta K_2(r) \left( 1 + o \left( \exp \left[ -\min \{m, z\} r \right] \right) \right)
\]
as \( r \to \infty \), so that the mass of the gauge field is 1 in our units; similarly one can identify the mass of the gauge field as \( m_A = \sqrt{\mathcal{L}(\ell+1)} \) in the Yang-Mills Higgs model \( [\mathcal{L}] \). What is more interesting, though, is that there is a limit to how large the mass of the Higgs field can be made: it cannot exceed twice the mass of the gauge field even if the mass parameter \( \sqrt{V''(1)} \) in the Lagrangian for the theory is large. Furthermore, the function multiplying the exponential undergoes an abrupt change in character as the parameter \( m \) increases past twice the mass of the gauge field.

Mathematically this arises because the differential equations

\[
-R''(r) - r^{-1} R'(r) + \left[ V''(1) + r^{-2} (S(r) + n)^2 - \hat{V} (1-R(r) \cdot (1-R(r))) \right] (R(r)-1)
\]

\[
= -r^{-2} (S(r) + n)^2
\]

and
\(- R''(r) - 2 \, r^{-1} R'(r) + \left[ V''(r) \, + \, \mathcal{L}(\ell+1) \, r^{-2} \, (S(r) + \eta) \right] \left( R(r) - 1 \right) \nabla \left( 1 - R(r) \right) \left( 1 - R(r) \right) \right) \left( R(r) - 1 \right) = - \mathcal{L}(\ell+1) \, r^{-2} \left( S(r) + \eta \right)^2 \) satisfied by \( R \) in the Abelian Higgs model and Yang-Mills Higgs model respectively are approximate modified Bessel equations with inhomogeneous terms. These inhomogeneous terms decay exponentially, with twice the gauge field's mass as inverse characteristic length, so that if \( m = [V''(1)]^{\frac{1}{2}} \) is larger than twice the gauge field's mass the subdominant homogeneous solution decays faster than the inhomogeneous term; in this case the asymptotics of the solution \( R - 1 \) is determined not by \( [V''(1)]^{\frac{1}{2}} \) but by the mass of the gauge field.

On the other hand, physical insight into this effect is gained by examining the heuristic arguments behind the predictions of spontaneous symmetry breaking and the Higgs phenomenon in these models (see [Ta]). Consider solutions \( \phi \) and \( A \) of the field equations whose action

\[ \mathcal{F}(\phi, A) = \int \left\{ \frac{1}{2} \| F(A) \|^2 + \frac{1}{2} \| \nabla \phi \|^2 + V(|\phi|) \right\} \]

is finite. Because of the potential term in \( \mathcal{F} \) we expect (and for isotropic solutions we have shown) that \( V(|\phi(x)|) \to 0 \) as \( |x| \to \infty \), i.e. the Higgs field approaches the manifold of classical vacua (the set of \( \phi_1 \in W \) such that \( V(\|\phi_1\|) = 0 \)) for the potential \( V \). The analogous phenomenon that is expected to occur in the corresponding quantum field theory is that the one-point function \( \langle \phi(x) \rangle \) does not vanish, but rather is some non-zero constant \( \hat{\phi} \in W \), so that there is spontaneous breaking of the invariance of
the theory under the symmetry group $G$ of the Lagrangian. One is then led to replace the dynamical variable $\phi$ by the field $\bar{\phi}$ defined $\phi = \bar{\phi} + \hat{\phi}$, where $V(\bar{\phi} + \hat{\phi}) = 0$; more generally one writes $\phi = U(g)(\bar{\phi} + \hat{\phi})$ and $A = g^{-1} \tilde{A} g^{-1} + g d g^{-1}$, with $g$ a $G$-valued function on space-time. By suitably choosing $g$ (i.e. choosing a gauge) the action becomes

$$\mathcal{F}(\tilde{\phi}, \tilde{A}) = \int \left\{ \frac{1}{2} \| F(\tilde{A}) \|^2 + \frac{1}{2} \| \tilde{\phi} \|^2 + V(\| \tilde{\phi} + \hat{\phi} \|) + \langle U(\tilde{A}) \bar{\phi} | U(\tilde{A}) \hat{\phi} \rangle + \frac{1}{2} \| U(\tilde{A}) \hat{\phi} \|^2 \right\}.$$

The extra terms $\frac{1}{2} \| U(\tilde{A}) \phi \|^2$ and $\langle U(\tilde{A}) \bar{\phi} | U(\tilde{A}) \hat{\phi} \rangle$ are interpreted, respectively, as a mass term for the gauge field and as an interaction term coupling the Higgs and gauge fields, which is represented by the vertex shown in Fig. 3. In the Abelian Higgs case, for example, we have written

$$\phi(x, y) = e^{i n \theta} (\tilde{R}(r) + 1)$$

and

$$A(x, y) = \tilde{S}(r) d\theta - n d\theta$$

(at least for $r \geq 2$, where $\tilde{R}(r) = 1$ and $\tilde{S}(r) = -n$); the mass and interaction terms become $\frac{1}{2} S^2$ and $\tilde{S}^2 \tilde{R}$ respectively. In particular it is the term $\tilde{S}^2 \tilde{R}$, i.e. the interaction term, that gives rise to the inhomogeneous side of the classical equations for $\tilde{R} = R - 1$. In this way the fact that the classical mass of the Higgs field cannot exceed twice the mass of the gauge field is directly related to the decay mode of Fig. 3 in which a Higgs particle decays into two gauge particles.
Finally, let us comment on other questions that may be asked about solutions of classical gauge field theories. One may ask whether our isotropic solutions are stable with respect to variations which are not spherically symmetric. In the Abelian Higgs model with the quartic double well as the potential, for example, it is expected ([Bo], [Ti]) that a vortex is stable if \( m < 1 \) and unstable if \( m > 1 \), since \( m = 1 \) corresponds to the transition from type I to type II superconductivity; for more general potentials the stability should be governed by the value \( V''(0) \). Stability is related to whether vortex and monopole solutions with multiple centers exist. In [Ti] it is shown that multi-vortices exist if the potential is the quartic double well with \( m = 1 \), while for \( m \neq 1 \) multi-vortices are expected to either coalesce at a point or disperse to infinity. Similarly, multi-monopoles exist in the Prasad-Sommerfield limit \( m \rightarrow 0 \), as shown in [T3]. For more general potentials one can also ask whether monopole solutions exist with higher monopole number. The answers to these questions are important to our understanding of the structure of gauge theories.
Appendix I

In this appendix we very briefly discuss the geometric picture of gauge fields. (For more details refer to [DM]. Our general reference for differential geometry is [KN]; much of our terminology is given a more elementary presentation in [CDD].) Given a manifold $M$ (called the base manifold) and a manifold $F$ (called the fiber) we can form what is called the trivial bundle $B = M \times F$ over $M$ with fiber $F$, which we regard as being equipped with the projection map $\pi_M : B \to M$ taking the point $(m,f) \in B$ to the point $m \in M$. More generally a bundle over $M$ with fiber $F$ is a manifold $B$, together with a map $\pi : B \to M$, that is locally isomorphic to the trivial bundle $M \times F$ over $M$ with fiber $F$; that is to say, $B$ is obtained by patching together copies of $M \times F$, perhaps with twists. (An easily visualized example is shown in Fig. 4. Here $B$, the Möbius strip, is a bundle over $M$, which is the circle $S^1$, with fiber $F = [0,1]$; this illustrates what we mean by "twists".) Bundles over $M$ with fiber $F$ are important because they enable us to generalize the notion of a function from $M$ to $F$. Given any map $s : M \to F$ define the map $\sigma : M \to M \times F$ by $\sigma(m) = (m,s(m))$ for $m \in M$; note that the composite map $\pi_M \circ \sigma$ is the identity map $1_M$ from $M$ to $M$, and that any map $\sigma : M \to M \times F$ such that $\pi_M \circ \sigma = 1_M$ is determined in this way by a unique map $s : M \to F$. We are thus motivated to generalize the notion of a map from $M$ to $F$ by defining a section of a bundle $\pi : B \to M$ over $M$ with fiber $F$ to be a map $\sigma : M \to B$ such that $\pi \circ \sigma = 1_M$. For example, the space $TM$ of vectors tangent to an $N$-dimensional manifold $M$, when equipped with the map $\pi_M : TM \to M$ which takes a tangent vector at $m \in M$ to the point $m$, is a bundle over $M$ with fiber $R^N$ called the tangent bundle of $M$; a section of this bundle is a vector field over $M$. For certain manifolds, such as the 2-dimensional sphere $S^2$, the tangent bundle may be non-trivial, i.e. twisted, so that vector fields are better understood in
terms of sections than maps from the manifold to $\mathbb{R}^N$.

In the special case when the fiber is a Lie group $G$ the trivial bundle $P = M \times G$ has extra structure: $G$ acts as a group of transformations on $P$ by defining $g_1 \in G$ to transform $(m, g_2) \in P$ into $(m, g_2 g_1) \in P$. A manifold $P$ on which $G$ acts is called a principal bundle with structure group $G$ if it is a bundle with fiber $G$ and the action of $G$ on $P$ is locally the same as that on the trivial principal bundle $M \times G$. If $G$ also acts on some manifold $F$ then there is a naturally associated bundle $P \times_G F$ over $M$ with fiber $F$ which is twisted in accordance with the twisting of $P$. In the application to gauge theories the Lie group $G$ is the group of symmetries, and the fields are sections of bundles associated with some principal bundle with structure group $G$.

It is important in our application to gauge theories that we be able to differentiate the fields (i.e. sections of associated bundles). If the bundle is trivial there is a canonical way to do this. Namely, if the section $\sigma : M \to M \times F$ is of the form $\sigma = (1_M, s)$ for some map $s : M \to F$, then we define the derivative $d\sigma : TM \to M \times TF$ to be the map $(\pi_{TM}, ds) : TM \to M \times TF$. For more general bundles, though, there is no corresponding prescription for defining the derivative of a section, even though bundles are locally isomorphic to trivial bundles, since the above definition of the derivative depends critically on the identification of a horizontal direction (along the base manifold $M$). To define the derivative one must equip the bundle with extra structure that identifies a horizontal direction at each point $p \in P$. This extra structure is called a connection for the bundle; in gauge theories the connection is called the gauge field.

If a principal bundle is equipped with a connection then its associated bundles inherit connections in a natural way. Thus we regard a gauge field as a connection in some principal bundle; the gauge field then defines a
type of differentiation, called covariant differentiation, for sections of associated bundles. To see what is required to specify a connection consider Fig. 5. The space $T_pP$ of tangent vectors to $P$ at $p \in P$ is locally isomorphic to the product $T_{\pi(p)}M \times \mathfrak{g}$ of the space $T_{\pi(p)}M$ of tangent vectors to $M$ at $\pi(p) \in M$ with the Lie algebra $\mathfrak{g}$ of $G$, which is the space $T_{eG}$ of tangent vectors to $G$ at the identity element $e \in G$.

To identify a horizontal direction at $p \in P$ is to pick a vector subspace $H_p$ of $T_{\pi(p)}M \times \mathfrak{g}$ that is supplementary to $\mathfrak{g}$ (so that $T_{\pi(p)}M \times \mathfrak{g} \oplus H_p \oplus \mathfrak{g}$). This subspace is conveniently parametrized by the projection

$$A_p : T_{\pi(p)}M \times \mathfrak{g} \cong H_p \oplus \mathfrak{g} \rightarrow \mathfrak{g}$$

along $H_p$. The map $p \mapsto A_p$ gives rise to a map $A:TP \rightarrow \mathfrak{g}$; such a map, if it satisfies certain natural requirements, is a connection. The covariant derivative associated to $A$ is given as follows: a section $\sigma$ of $P \times G^F$ given locally by the map $s:M \rightarrow F$ gives rise to a map

$$d_A s = ds + A \cdot s$$

from $TM$ to $TF$ (where $A$, given locally by the above projection, takes values in the Lie algebra $\mathfrak{g}$ of $G$ and hence acts on members of $F$ to give tangent vectors of $F$); this map locally specifies the covariant derivative $d_A \sigma$. The requirements made on the map $A:TP \rightarrow \mathfrak{g}$ for it to be a connection are such that this local definition of $d_A \sigma$ actually defines a global section of the bundle $L(TM, P \times G^TF)$ of maps from $TM$ to $P \times G^TF$.

Note that the connection $A$ is not a section of any bundle, although it can be described locally in terms of Lie algebra-valued $1$-forms on $M$. However, we may still form its covariant derivative
\[ F(A) = \int_A A = \int A + \frac{1}{2} \int A \wedge A, \]
called the curvature of \( A \). (Here \([A, A]\) involves a wedge product as well as a lie bracket; the \( \frac{1}{2} \) appears because \( A \) is not a section of a bundle.) The curvature is a section of an associated bundle (the adjoint bundle), and it measures the deviation of \( A \) from the trivial connection for the trivial bundle described above, which has zero curvature and hence is called a flat connection. In gauge theories the curvature has the significance of being the field strength for the force mediated by the gauge particles.

Let us see how to incorporate gauge fields into a field theory which has an invariance group \( G \). Suppose the action for the theory is

\[ S(\phi) = \int_M L(\phi, d\phi) \]

Suppose further that the field \( \phi \) transforms under the representation \( g \to U(g) \) of \( G \), and that \( S \) is globally invariant under \( G \)(i.e. \( S(U(g)\phi) = S(\phi) \) for all \( g \in G \)). Then the requirement that the action be locally invariant under \( G \) (i.e. \( S(U(g)\phi) = S(\phi) \) for each map \( g: M \to G \)) leads one [YM] to the theory with action

\[ \mathcal{S}(\phi, A) = \int_M \left\{ \frac{1}{2} \|F(A)\|^2 + L(\phi, dA) \right\} , \]

where \( A \) is a new dynamical variable that can be thought of as a connection in a principle bundle over \( M \) with structure group \( G \). As an example of this procedure consider a theory of a complex-valued field \( \phi \) whose action \( S \) is left invariant by the transformation \( \phi \to e^{i\alpha} \phi \) for \( \alpha \in [0, 2\pi[ \).

Then we obtain a gauge theory with invariance group \( U(1) \) and action
\[ \mathcal{S}(\phi, A) = \int_M \left\{ \frac{1}{2} \|A\|^2 + \mathcal{L}(\phi, (d+iA)\phi) \right\}, \]

where the gauge field is written \( iA \) (so that \( A \) is real-valued, the Lie algebra of \( U(1) \) being \( i\mathbb{R} \)). This is the theory of a charged scalar field interacting with an electromagnetic field, as obtained by the usual minimal coupling prescription.

The usefulness of differential-geometric language for gauge theories is apparent when one studies solutions of the (classical) equations of motion. For instance the Chern-Weil theorem in the theory of bundles can be invoked to classify gauge fields in terms of integral invariants of the corresponding bundle. This theorem states that certain polynomials in \( \mathcal{F}(A) \) give differential forms on \( M \), so-called characteristic classes, that depend only on the structure of the principal bundle \( P \), not on \( A \) directly, and moreover that give integers when integrated over \( M \) (at least if \( M \) is compact). For example, the Chern classes \( c_j \), given by

\[
\det \left( 1 + \frac{i}{2\pi} \mathcal{F}(A) \right) = 1 + c_1 + c_2 + \ldots + c_{\left[ \frac{\dim M}{2} \right]}
\]

are such differential forms in case \( G \subseteq U(N) \) for some \( N \). In particular we have:

\[
c_1 = \frac{i}{2\pi} \, d(\, iA) = -\frac{1}{2\pi} \, dA
\]

in case \( G = U(1) \), and

\[
c_2 = -\left( 32\pi^4 \right)^{-1} \kappa(\mathcal{F}(A), \mathcal{F}(A)),
\]
where $\kappa$ is the Killing form of the Lie algebra of $SU(2)$, in case $G = SU(2)$; the integral of $c_1$ is known as the vortex number, and the integral of $c_2$ is known as the topological charge. Related invariants, the secondary characteristic classes of Chern and Simons, can also be used to classify singular gauge fields according to the topological charge concentrated at their singularities [P].

In another vein, a restricted class of solutions, the self-dual solutions (known as instanton solutions), of the equations of motion for the pure Yang-Mills (i.e. $SU(2)$) gauge theory over the four dimensional sphere has been completely described by invoking deep theorems on the classification of algebraic bundles [AHS]. Some progress has been made toward showing that all solutions belong to this restricted class [T2].
Appendix II

In this appendix we derive some properties of modified Bessel functions which we use above. Let us define the functions $i_v$ and $k_v$ on $\mathbb{R}_+$, for $\Re v > -\frac{1}{2}$, by

$$i_v(r) = e^r \left[ \Gamma \left( v + \frac{1}{2} \right) \right]^{-1} \int_0^{2r} ds \ e^{-s} s^{-v - \frac{1}{2}} (1 - \frac{s}{2r})^{v - \frac{1}{2}}$$

and

$$k_v(r) = e^{-r} \left[ \Gamma \left( v + \frac{1}{2} \right) \right]^{-1} \int_0^{2r} ds \ e^{-s} s^{-v - \frac{1}{2}} (1 + \frac{s}{2r})^{v - \frac{1}{2}}$$

$i_v$ and $k_v$ satisfy the differential equation

$$-w''(r) + \left[ 1 + (v^2 - \frac{1}{4}) r^{-2} \right] w(r) = 0,$$

as may be verified directly, and are related to the modified Bessel functions $I_v$ and $K_v$ by [BMP]

$$I_v(r) = (2\pi r)^{-\frac{1}{2}} i_v(r)$$

and

$$K_v(r) = (\pi/2r)^{\frac{1}{2}} k_v(r).$$

Using the inequality

$$\left| (1 + x)^{\alpha} - 1 \right| \leq |\alpha| |x| \cdot \max \left\{ 1, (1 + x)^{\Re \alpha} \right\}$$
(for $1 + x \geq 0$) one finds that

$$\left| e^r k_v(r) - 1 \right| \leq \left| \Gamma(v + \frac{1}{2}) \right|^{-1} \int_0^\infty ds \ e^{-s} \ |s^{v - \frac{1}{2}}| \cdot \left| v - \frac{1}{2} \right| \cdot \frac{s}{2r} \ \max \left\{ 1, \left( \frac{1 + \frac{5}{2r}}{\Delta} \right)^{\frac{1}{2}} \right\}$$

$$\leq r^{-1} \cdot \frac{1}{2} \left| v - \frac{1}{2} \right| \left| \Gamma(v + \frac{1}{2}) \right|^{-1} \int_0^\infty ds \ e^{-s} \ |s^{v - \frac{1}{2}}| \ \max \left\{ 1, \left( \frac{1 + \frac{5}{2r}}{\Delta} \right)^{\frac{1}{2}} \right\}$$

if $r \geq r_o$, so $k_v(r) = e^{-r}(1 + O(r^{-1}))$ as $r \to \infty$. Moreover,

$$e^r \left( k'_v(r) + \frac{1}{2} k_{v'}(r) \right) = -r^{-2} \cdot \frac{1}{2} (v - \frac{1}{2}) \left| \Gamma(v - \frac{1}{2}) \right|^{-1} \int_0^\infty ds \ e^{-s} \ s^{v - \frac{3}{2}} \left( 1 + \frac{s}{2r} \right)^{v - \frac{1}{2}}$$

we conclude that $k_v'(r) = -k_v(r) \cdot (1 + O(r^{-2}))$ as $r \to \infty$, just as above.

In like manner one can prove that $i_v(r) = e^r(1 + O(r^{-1}))$ and

$$i_v'(r) = i_v(r) \cdot (1 + O(r^{-2}))$$ as $r \to \infty$.

To see that $k_v$ is real-valued if $v^2$ is real, note that $\Re k_v$ satisfies the same differential equation and exhibits the same asymptotic behavior as $k_v$. Since both $\Re k_v$ and $k_v$ are subdominant to $i_v$ at infinity it follows that $\Re k_v = k_v$.

Finally, the asymptotics of $i_v$ at the origin may be derived from the formula

$$i_v(r) = e^r (2r)^{v - \frac{1}{2}} \left| \Gamma(v + \frac{1}{2}) \right|^{-1} \int_0^1 dt \ e^{-2rt} \cdot t^{v - \frac{1}{2}} (1 - t)^{v - \frac{1}{2}}$$

obtained by changing the variable $s$ to $2rt$ in the definition of $i_v$.

Indeed by a standard formula for the beta function and by Legendre's duplication formula

$$\left[ \Gamma(v + \frac{1}{2}) \right]^{-1} \int_0^1 dt \ t^{v - \frac{1}{2}} (1 - t)^{v - \frac{1}{2}} = \left[ \Gamma(2v + 1) \right]^{-1} \cdot \Gamma(v + \frac{1}{2})$$

$$= \left[ 2^{2v} \cdot \Gamma(v + 1) \cdot \Gamma(v + \frac{1}{2}) \right]^{-1} \cdot \Gamma(v + \frac{1}{2}) \cdot \Gamma(v + \frac{1}{2})$$
so that \( i_\nu(r) = 2\pi^{\frac{\nu}{2}} [\Gamma(\nu + 1)]^{-1} \left(\frac{r}{2}\right)^{\nu + \frac{1}{2}} \cdot (1 + o(r)) \) and

\[ i'_\nu(r) = \pi^{\frac{\nu}{2}} [\Gamma'(\nu + 1)]^{-1} \left(\frac{r}{2}\right)^{\nu - \frac{1}{2}} \cdot (1 + o(r)) \text{ as } r \to 0. \]
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Figure 2
$F = [0,1]$ \hspace{1cm} B = \text{Möbius strip}$

\[
\begin{array}{c}
\downarrow \\
\Pi \\
\downarrow \\
\end{array}
\]

$M = \text{circle}$