The behavior at infinity of isotropic vortices and monopoles [a)]

Bradley Plohr
The Rockefeller University, New York, New York 10021

(Received 6 April 1981; accepted for publication 18 May 1981)

We derive detailed asymptotic formulae for the behavior at infinity of isotropic vortex solutions of the abelian Higgs model and monopole solutions of the Yang–Mills Higgs model. In particular we find that the classical mass of the Higgs field is the smaller of m and twice the mass of the gauge field, where $(mc/\hbar)^2$ is the curvature of the Higgs self-interaction potential at the classical vacuum.

PACS numbers: 03.50.Kk, 11.10.Jj, 11.10.Lm

1. INTRODUCTION

In this paper we present an analysis of the asymptotic behavior at large distances of certain isotropic solutions of classical gauge field equations, namely the Nielsen–Olesen vortex solutions of the abelian Higgs model in two-dimensional Euclidean space, and the t' Hooft–Polyakov monopole solutions of the Yang–Mills Higgs model in three-dimensional Euclidean space. The behavior at infinity of these classical solutions field indicates that the Higgs mechanism of symmetry breaking is operative in the quantized versions of these models, and the exact asymptotics determines the classical approximation of the masses of the corresponding quantum particles. We find that the mass of the Higgs particle is $m_\phi = \min\{\hbar/c\cdot[V''(R_\infty)]^{1/2}, 2m_A\}$, where V is the Higgs self-interaction potential, R_∞ is the asymptotic value of the norm of the Higgs field (i.e., the classical vacuum), and m_A is the mass of the gauge field. A physical interpretation of this result is given in the Conclusion.

2. THE ABELIAN HIGGS MODEL

The abelian Higgs model describes a charged scalar Higgs field which is self-coupled via a potential V and which interacts with an abelian [i.e., U(1)] gauge field in two-dimensional Euclidean space. Thus the Higgs field ϕ is a complex-valued function on $\mathbb{R}^2$, and the gauge field may be written as iA, where A is a real-valued 1-form on $\mathbb{R}^2$. [We represent the Lie algebra of U(1) as $i\mathbb{R}$.] The potential V is assumed to be twice continuously differentiable, nonnegative, and symmetric about the origin; furthermore we assume that V has a zero at $R_\infty > 0$ such that $V''(R_\infty) > 0$ and $V(\hat{R}) > 0$ if $\hat{R} < R_\infty$. (See Fig. 1.)

Using units in which $R_\infty = 1$ and $e/\hbar c = 1$ (where the charge of the Higgs field is taken to be $-e < 0$), the Euclidean action for the abelian Higgs model is

$$\mathscr{A}(\phi,A) = \int_{\mathbb{R}^2} d^2x\,\{\tfrac{1}{2}\|F(A)\|^2 + \tfrac{1}{2}\|d_A\phi\|^2 + V(\|\phi\|)\}; \quad (2.1)$$

$F(A) = dA$ is the field strength of A and $d_A\phi = d\phi + iA\phi$ is the covariant derivative of ϕ. The critical points of $\mathscr{A}$ formally satisfy the equations

$$d_A^* d_A\phi + V'(\|\phi\|)\phi/\|\phi\| = 0, \quad (2.2a)$$

$$d^* dA + (1/2i)(\bar{\phi} d_A\phi - \overline{\bar{\phi} d_A\phi}) = 0 \quad (2.2b)$$

(here * denotes the formal adjoint with respect to the appropriate inner product, and $^-$ denotes complex conjunction). These are the abelian Higgs equations.

We wish to consider vortex solutions of Eqs. (2.2a) and (2.2b) characterized by having finite Euclidean action (2.1) and by exhibiting a classical version of the Higgs symmetry-breaking mechanism, viz.

$$\lim_{x\to\infty} \|\phi(x)\| = 1.$$

Following Nielsen and Olesen[1] we look for isotropic solutions of the form

$$\phi(x,y) = R(r)\exp(in\theta) \quad (2.3a)$$

and

$$A(x,y) = S(r)\,d\theta, \quad (2.3b)$$

where n is an integer, and r and θ are polar coordinates defined by $x = r\cos\theta$ and $y = r\sin\theta$. It can be shown[2] that for every integer n there exist Higgs and gauge fields ϕ and A of the form (2.3) which are twice continuously differentiable and satisfy (2.2) throughout $\mathbb{R}^2$, and for which the Euclidean action is finite and $\lim_{|x|\to\infty}\|\phi(x)\| = 1$. (The integer n is known as the vortex number.) In this paper we study the asymptotic behavior of such fields at large distances.

As one may easily verify, the real-valued functions R and S on $\mathbb{R}_+ = \{r\in\mathbb{R}\,|\,r>0\}$ satisfy the coupled pair of nonlinear differential equations

$$-R''(r) - r^{-1}R'(r) + r^{-2}(S(r)+n)^2R(r) + V'(R(r)) = 0 \quad (2.4a)$$

and

$$-S''(r) + r^{-1}S'(r) + R^2(r)(S(r)+n) = 0. \quad (2.4b)$$

Furthermore,

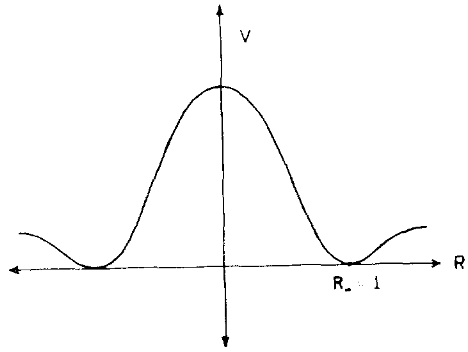

FIG. 1. A typical Higgs field self-interaction potential.

[a)]Supported in part by NSF-PHY-80-01979. This work represents part of the author's Ph.D. dissertation presented to the Department of Physics, Princeton University in August, 1980.

$$F(R,S) = \int_0^\infty r\,dr\,[\tfrac{1}{2}r^{-2}[S'(r)]^2 + \tfrac{1}{2}[R'(r)]^2 + \tfrac{1}{2}r^{-2}R^2(r)(S(r)+n)^2 + V(R(r))] \qquad (2.5)$$

is finite since $F(R,S)$ coincides with $(2\pi)^{-1}\mathscr{A}(\phi,A)$; and we may assume that

$$\lim_{r\to\infty} R(r) = 1 \qquad (2.6)$$

because $\lim_{|x|\to\infty}\|\phi(x)\| = 1$ and we can use a (global) gauge transformation to ensure that $R(r)$ is positive for large r. In Sec. 3 we will prove

Theorem 1: Suppose that R and S satisfy (2.4)–(2.6) let $m = [V''(1)]^{1/2}$. Then there exist constants α and β such that

$$S(r) = -n + \beta r K_1(r)\cdot(1 + o(\exp[-\min\{m,2\}r]))$$

and

$$S'(r) = -(S(r)+n)\cdot(1-(2r)^{-1} + O(r^{-2}))$$

as $r\to\infty$, and such that:

(a) if $m<2$,

$$R(r) = 1 - \alpha K_0(mr).(1 + o(\exp(-mr)) - [m^2-4]^{-1}\beta^2[K_1(r)]^2(1+o(1))$$

and

$$R'(r) = -m(R(r)-1)\cdot(1+(2mr)^{-1}+O(r^{-2}));$$

(b) if $m=2$,

$$R(r) = 1 - \tfrac{1}{2}\beta^2 r[K_1(r)]^2\cdot(1+o(1))$$

and

$$R'(r) = -2(R(r)-1)\cdot(1+o(1));$$

(c) if $m>2$,

$$R(r) = 1 - [m^2-4]^{-1}\beta^2[K_1(r)]^2\cdot(1+o(1))$$

and

$$R'(r) = -2(R(r)-1)\cdot(1+o(1))$$

as $r\to\infty$.

From this result follows the decay properties of gauge-invariant quantities of physical importance, such as $\|\phi\| = |R|$, $\bar{\phi} d_A\phi = R\cdot R'\,dr + iR^2\cdot(n+S)\,d\theta$, and $dA = r^{-1}S'\,dr\wedge r d\theta$. In particular, the mass of the gauge field is $m_A = \hbar/c\cdot e/\hbar c\cdot R_\infty$ and the mass of the Higgs boson is $m_\phi = \min\{\hbar/c\cdot[V''(R_\infty)]^{1/2}, 2m_A\}$.

3. PROOF OF THEOREM 1

In order to determine the asymptotics of our fields we will find it convenient to work with the shifted fields u and v defined by

$$u(r) = r^{1/2}(1-R(r)) \qquad (3.1a)$$

and

$$v(r) = r^{-1/2}(n+S(r)) \qquad (3.1b)$$

instead of with R and S. Given that R and S satisfy Eqs. (2.4a) and (2.4b), u and v satisfy the differential equations

$$-u''(r) + [V''(1) - r^{-2}/4 - \widehat{V}(r^{-1/2}u(r))r^{-1/2}u(r) + r^{-1}v^2(r)]u(r) = r^{-1/2}v^2(r) \qquad (3.2a)$$

and

$$-v''(r) + [1 + \tfrac{3}{4}r^{-2} - 2r^{-1/2}u(r) + r^{-1}u^2(r)]\,v(r) = 0, \qquad (3.2b)$$

where we have written $V'(1-\widetilde{R}) = -V''(1)\widetilde{R} + \widehat{V}(\widetilde{R})\widetilde{R}^2$. Note that $\widehat{V}(\widetilde{R}) = O(1)$ as $\widetilde{R}\to 0$ by Taylor's theorem.

In the proof of Theorem 1 we will show that $u(r)$ and $v(r)$ vanish as $r\to\infty$. We will then be in a position to apply the following lemma, which shows that solutions of differential equations of the form of Eqs. (3.2a) and (3.2b) decay exponentially at infinity, assuming that they vanish at infinity. The proof of the lemma is based on an elementary form of the maximum principle.[3]

Lemma 3.1: Suppose w is a C^2 solution on $\mathbb{R}_+$ of the differential equation

$$-w''(r) + (\kappa^2 + f(r))w(r) = g(r),$$

where f and g are continuous functions on $\mathbb{R}_+$ *and* $\kappa>0$. If w and f vanish at infinity, and

$$g(r) = O(\exp(-\lambda r))$$

as $r\to\infty$ for some $\lambda>0$, then for every positive $\epsilon<1$

$$w(r) = O(\exp[-\min\{\kappa(1-\epsilon)^{1/2},\lambda\}r])$$

as $r\to\infty$.

Proof: Fix $0<\epsilon<1$. By hypothesis there is an $r_\epsilon\in\mathbb{R}_+$ and a constant γ_ϵ such that $\frac{1}{2}\epsilon\kappa^2 + f(r)\geqslant 0$ for all $r\geqslant r_\epsilon$ and such that $|g(r)|\leqslant\gamma_\epsilon\exp(-\lambda r)$ for all $r\geqslant r_\epsilon$. Define $\kappa_\epsilon = \kappa(1-\frac{1}{2}\epsilon)^{1/2}$, $f_\epsilon = \frac{1}{2}\epsilon\kappa^2 + f$, and $\mu_\epsilon = \min\{\kappa(1-\epsilon)^{1/2},\lambda\}$. Thus

$$-w''(r) + (\kappa_\epsilon^2 + f_\epsilon(r))w(r) = g(r),$$

with $f_\epsilon(r)\geqslant 0$ for all $r\geqslant r_\epsilon$ and $g(r)\leqslant\gamma_\epsilon\exp(-\mu_\epsilon r)$ for all $r\geqslant r_\epsilon$.

Define the function $\widetilde{w}$ by

$$\widetilde{w}(r) = \rho_\epsilon\exp(-\kappa_\epsilon r) + [\kappa_\epsilon^2 - \mu_\epsilon^2]^{-1}\gamma_\epsilon\exp(-\mu_\epsilon r) - w(r),$$

where $\rho_\epsilon\geqslant 0$ is chosen so that $\widetilde{w}(r_\epsilon)\geqslant 0$. Then one easily verifies that

$$-\widetilde{w}''(r) + [\kappa_\epsilon^2 + f_\epsilon(r)]\widetilde{w}(r) = f_\epsilon(r)[\widetilde{w}(r) + w(r)] + \gamma_\epsilon\exp(-\mu_\epsilon r) - g(r)\geqslant 0$$

for all $r\geqslant r_\epsilon$.

We will show that $\widetilde{w}(r)\geqslant 0$ if $r\geqslant r_\epsilon$. Suppose to the contrary that $\widetilde{w}(r_1)<0$ for some $r_1>r_\epsilon$; then because $\widetilde{w}(r_\epsilon)\geqslant 0$ and $\widetilde{w}(r)\to 0$ as $r\to\infty$ there exists an $r_{\min}>r_\epsilon$ at which $\widetilde{w}$ attains a strictly negative minimum. But then

$$\widetilde{w}''(r_{\min})\leqslant[\kappa^2 + f_\epsilon(r_{\min})]\widetilde{w}(r_{\min})<0,$$

contradicting the fact that $\widetilde{w}$ attains a minimum at $r_{\min}$; thus $\widetilde{w}(r)\geqslant 0$ for all $r\geqslant r_\epsilon$. This inequality implies the bound

$$\widetilde{w}(r)\leqslant \text{const}\ \exp(-\mu_\epsilon r)$$

for all $r\geqslant r_\epsilon$, since $\kappa_\epsilon\geqslant\mu_\epsilon$. Because we may just as well apply this argument to $-w$, we deduce that $w(r) = O(\exp(-\mu_\epsilon r))$ as $r\to\infty$.

By appealing to this lemma we will see that u and v satisfy equations which are modified Bessel equations, except for perturbations which are exponentially small at infinity. Therefore we will compare u and v to the modified Bessel functions that approximately solve Eqs. (3.2a) and (3.2b) to obtain more precise results about their asymptotic behavior. Let K_ν denote the usual[4] modified Bessel function of order ν that is subdominant at infinity, and define k_ν on $\mathbb{R}_+$ by

$$k_\nu(r) = (2r/\pi)^{1/2} K_\nu(r);$$

$w = k_\nu$ solves the differential equation

$$-w''(r) + (1 + [\nu^2 - \tfrac{1}{4}] r^{-2}) w(r) = 0.$$

The following properties[4] of k_ν are used below: if ν^2 is real, then k_ν is a real-valued function such that

$$k_\nu(r) = \exp(-r)\cdot(1 + O(r^{-1}))$$

and

$$k'_\nu(r) = -k_\nu(r)\cdot(1 + O(r^{-2}))$$

as $r \to \infty$.

Lemma 3.2 is a typical application of the variation-of-parameters technique from the theory of ordinary differential equations. It is a slightly refined form of the WKB approximation.

Lemma 3.2: Consider the differential equation

$$-w''(r) + (\kappa^2 + [\nu^2 - \tfrac{1}{4}] r^{-2} + h(r)) w(r) = 0,$$

where h is a continuous function on $\mathbb{R}_+$, ν^2 is real, and $\kappa > 0$. If $|h|$ is integrable at infinity then there exist C^2 solutions w^+ and w^- of this equation such that

$$w^\pm = \exp(\pm \kappa r)\cdot(1 + o(1))$$

and

$$[w^\pm]'(r) = \pm \kappa w^\pm(r)\cdot(1 + o(1))$$

as $r \to \infty$. If in addition $h(r) = O(\exp(-\mu r))$ as $r \to \infty$, where $\mu > 0$, then

$$w^-(r) = k_\nu(\kappa r)\cdot(1 + O(\exp(-\mu r))$$

and

$$[w^-]'(r) = -\kappa w^-(r)\cdot(1 + O(r^{-2})).$$

Proof: For notational convenience we will assume that $\kappa = 1$; the general case can be reduced to this case by rescaling the independent variable. We first seek a solution w^- of our equation which is of the form $w^- k_\nu \cdot z$, where $z(r) \to 1$ as $r \to \infty$. Motivated by standard tricks used to solve ordinary differential equations we proceed as follows. Suppose we can find a C^2 solution z of the Volterra integral equation

$$z(r) = 1 + \int_r^\infty dr'\, K(r,r') h(r') z(r'),$$

where

$$K(r,r') = [k_\nu(r')]^2 \int_r^{r'} dr''\, [k_\nu(r'')]^{-2}.$$

Then z satisfies the differential equation

$$-z''(r) - 2k'_\nu(r)[k_\nu(r)]^{-1} z'(r) + h(r) z(r) = 0,$$

so that the product $w^- = k_\nu \cdot z$ satisfies our equation. It is thus of interest to find z; we will construct a solution of the Volterra integral equation in the usual manner, namely by proving that its Neumann series converges.

Throughout the following fix $r_0 \in \mathbb{R}_+$ such that $k_\nu(r) > 0$ if $r \geqslant r_0$. It is easy to check that there is a constant ρ such that $0 \leqslant K(r,r') < \tfrac{1}{2}\rho$ and $0 \leqslant -(\partial K/\partial r)(r,r') \leqslant \rho$ for all $r' \geqslant r \geqslant r_0$. For later convenience let $H(r) = \tfrac{1}{2}\rho \int_r^\infty dr' |h(r')|$. Note that $H(r) = o(1)$ as $r \to \infty$ since $|h|$ is integrable at infinity; furthermore, $H(r) = o(\exp(-\mu r))$ as $r \to \infty$ in case $h(r) = o(\exp(-\mu r))$ as $r \to \infty$ by 1' Hôpital's rule.

Define $z_0 = 1$, and for nonnegative integers j define z_{j+1} inductively by

$$z_{j+1}(r) = \int_r^\infty dr' \quad K(r,r') h(r') z_j(r').$$

Then we find that $|z_j| \leqslant (j!)^{-1} H^j$ and $\tfrac{1}{2}|z'_j| \leqslant (j!)^{-1} H^j$ by a simple induction argument applied to the formulae for z_j and z'_j. From the first estimate we conclude that the series $\Sigma_{j=0}^\infty z_j$ converges uniformly on $]r_0, \infty[$ to a function z such that

$$|z(r) - 1| \leqslant \exp(H(r)) - 1 \leqslant \text{const } H(r)$$

for all $r > r_0$; the second estimate, along with the formula for z''_j, shows that z is C^2 on $]r_0, \infty[$ and that

$$\tfrac{1}{2}|z'(r)| \leqslant \exp(H(r)) - 1 \leqslant \text{const } H(r)$$

for all $r > r_0$. In addition, z is a solution of the Volterra integral equation, as follows by the Lebesgue dominated convergence theorem.

As a consequence $w^- = k_\nu \cdot z$ is a solution of our original equation with the properties that

$$\begin{aligned} w^-(r) &= k_\nu(r) z(r) = k_\nu(r)\cdot(1 + O(H(r))) \\ &= \exp(-r)\cdot(1 + o(1)) \end{aligned}$$

and

$$\begin{aligned} [w^-]'(r) &= k'_\nu(r) z(r) + k_\nu(r) z'(r) \\ &= -w^-(r)\cdot[1 + O(r^{-2}) + O(H(r))] \end{aligned}$$

as $r \to \infty$. We may construct a second solution w^+ by setting

$$w^+(r) = 2w^-(r) \int_{r_1}^r dr'\, [w^-(r')]^{-2}$$

for $r > r_1$ [where we choose $r_1 > r_0$ large enough that $w^-(r) \neq 0$ for all $r \geqslant r_1$]. That w^+, so defined, solves the same equation as does w^- is easily checked directly; the motivation for defining w^+ in this way comes from requiring that the Wronskian of w^+ and w^- be a constant, which is taken to be -2. The asymptotic behavior of w^+ may be determined as follows: by l'Hôpital's rule

$$\begin{aligned} \lim_{r\to\infty} w^+(r) w^-(r) &= \lim_{r\to\infty} \frac{2\int_{r_1}^r dr'\, [w^-(r')]^{-2}}{[w^-(r)]^{-2}} \\ &= \lim_{r\to\infty} \frac{2[w^-(r)]^{-2}}{-2[w^-]'(r)[w^-(r)]^{-3}} = 1, \end{aligned}$$

so that $w^+(r) = \exp(r)\cdot(1 + o(1))$ as $r \to \infty$; and

$$\begin{aligned} &\lim_{r\to\infty} [w^+]'(r)[w^+(r)]^{-1} \\ &\quad = \lim_{r\to\infty} [w^-]'(r)[w^-(r)]^{-1} + 2[w^-(r) w^+(r)]^{-1} = 1, \end{aligned}$$

so that $[w^+]'(r) = w^+(r)\cdot(1 + o(1))$ as $r \to \infty$. This completes the proof.

We now are in a position to prove Theorem 1. Define u and v through Eqs. (3.1a) and (3.1b). First of all, there is a strictly positive constant such that

$$V(R(r)) \geqslant \text{const } (1 - R(r))^2$$

for sufficiently large r; this is because $V(R) = V''(1) \times (1 - \hat{R})^2 + o(1 - \hat{R})^2$ as $\hat{R} \to 1$, where $V''(1) > 0$. Since $F(R,S)$ is finite we conclude that u is square-integrable at infinity. Since

$$u'(r) = -r^{1/2} R'(r) + \tfrac{1}{2} r^{-1} u(r)$$

the finiteness of $F(R,S)$ shows further that u' is square-integrable at infinity. Hence $(u^2)' = 2u\cdot u'$ is integrable at infinity, so that $\lim_{r\to\infty} u(r)$ exists; but this limit must vanish in order that u^2 be integrable at infinity. Since the third term

$$\int_0^\infty r\,dr\,\tfrac{1}{2}r^{-2}R^2(r)(S(r)+n)^2 = \int_o^\infty dr\,\tfrac{1}{2}(1-r^{-1/2}u(r))^2 v^2(r)$$

in $F(R,S)$ is finite it follows that v is square-integrable at infinity. Since

$$v'(r) = r^{-1/2}S'(r) - \tfrac{1}{2}r^{-1}v(r),$$

the finiteness of $F(R,S)$ guarantees that v' is also square-integrable at infinity. As before we conclude that v vanishes at infinity.

An examination of Eqs. (3.2a) and (3.2b) shows that we may apply Lemma 3.1 to v: given any positive $\epsilon < 1$,

$$v(r) = O(\exp[-(1-\epsilon)^{1/2}r])$$

as $r\to\infty$. Using this bound we may apply Lemma 3.1 to u

$$u(r) = O(\exp[-\min\{m,2\}(1-\epsilon)^{1/2}r])$$

as $r\to\infty$.

Again referring to Eqs. (3.2a) and (3.2b) we see that Lemma 3.2 applies to the equation satisfied by v. Since v vanishes at infinity it must be proportional to the subdominant solution constructed in Lemma 3.2, so there exists a constant β' such that

$$v(r) = \beta' k_1(r)\cdot(1 + o(\exp[-\min\{m,2\}(1-\epsilon)^{1/2}r]))$$

and

$$v'(r) = v(r)\cdot(1 + O(r^{-2}))$$

as $r\to\infty$.

Consider now the equation satisfied by u, which we will write as

$$-u''(r) + [m^2 - \tfrac{1}{4}r^{-2} + h(r)]\,u(r) = g(r).$$

Let u_+ and u_- denote the solutions of the corresponding homogeneous equation that are constructed in Lemma 3.2. Using u_+ and u_- we may construct a particular solution u_p of the equation satisfied by u: let

$$u_p(r) = (2m)^{-1}\int_{r_1}^r dr'\,u_-(r)u_+(r')g(r') + (2m)^{-1}\int_r^\infty dr'\,u_+(r)u_-(r')g(r')$$

for $r \geqslant r_1$. Then

$$-u_p''(r) + (m^2 - \tfrac{1}{4}r^{-2} + h(r))u_p(r) = -(2m)^{-1}W(r)g(r),$$

where the Wronskian $W(r) = u_+(r)u_-'(r) - u_-(r)u_+'(r) = -2m(1+o(1))$ by Lemma 3.2; but since $W' = 0$ we must have the $W = -2m$, so that u_p satisfies the correct equation. Because $u - u_p$ satisfies the homogeneous equation there exist constants α_+ and α_- such that $u = \alpha_+ u_+ + \alpha_- u_- + u_p$. We will determine the asymptotic behavior of u by examining u_p.

First of all,

$$\lim_{r\to\infty}\frac{(2m)^{-1}\int_\infty^r dr'\,u_-(r')g(r')}{g(r)[u_+(r)]^{-1}} = \lim_{r\to\infty}\frac{-2(m)^{-1}u_-(r)g(r)}{g(r)[u_+(r)]^{-1}\{(\ln g)'(r) - (\ln u_+)'(r)\}} = (2m)^{-1}\Big[m - \lim_{r\to\infty}(\ln g)'(r)\Big]^{-1}$$

by l'Hôpital's rule. But with $g(r) = r^{-1/2}v^2(r)$,

$$(\ln g)'(r) = -\tfrac{1}{2}r^{-1} + 2(\ln v)'(r) = -2(1 + \tfrac{1}{4}r^{-1} + O(r^{-2}))$$

as $r\to\infty$, so that the second term in u_p exhibits the behavior

$$(2m)^{-1}\int_r^\infty dr'\,u_+(r)u_-(r')g(r') = (2m)^{-1}(m+2)^{-1}g(r)\cdot(1+o(1))$$

as $r\to\infty$.

Suppose $m<2$. Then the first term in u_p may be written as

$$(2m)^{-1}\int_{r_1}^\infty dr'\,u_+(r')g(r')\cdot u_-(r) - (2m)^{-1}\int_r^\infty dr'\,u_-(r)u_+(r')g(r')$$

because $u_+(r)g(r)$ vanishes exponentially as $r\to\infty$. But

$$\lim_{r\to\infty}\frac{-(2m)^{-1}\int_r^\infty dr'\,u_+(r')g(r')}{g(r)[u_-(r)]^{-1}} = \lim_{r\to\infty}\frac{(2m)^{-1}u_+(r)g(r)}{g(r)[u_-(r)]^{-1}\{(\ln g)'(r) - (\ln u_-)'(r)\}} = (2m)^{-1}(m-2)^{-1}.$$

Thus there is a constant α'' such that

$$u_p(r) = \alpha'' u_-(r) + [m^2-4]^{-1}g(r)\cdot(1+o(1))$$

as $r\to\infty$. Since u vanishes at infinity it follows that there is a constant α' such that

$$u(r) = \alpha' k_0(mr)\cdot(1 + o(\exp[-m(1-\epsilon)^{1/2}r])) + [m^2-4]^{-1}r^{-1/2}[\beta' k_1(r)]^2\cdot(1+o(1))$$

as $r\to\infty$. If on the other hand $m>2$, then because $g\cdot[u_-]^{-1}$ grows (exponentially) at infinity,

$$\lim_{r\to\infty}\frac{(2m)^{-1}\int_{r_1}^r dr'\,u_+(r')g(r')}{g(r)[u_-(r)]^{-1}} = \lim_{r\to\infty}\frac{(2m)^{-1}u_+(r)g(r)}{g(r)[u_-(r)]^{-1}\{(\ln g)'(r) - (\ln u_-)'(r)\}} = (2m)^{-1}(m-2)^{-1},$$

so that

$$u(r) = \alpha_- k_0(mr)\cdot(1 + o(\exp[-2(1-\epsilon)^{1/2}r])) + [m^2-4]^{-1}r^{-1/2}[\beta' k_1(r)]^2\cdot(1+o(1)).$$

Lastly, in case $m=2$, $rg(r)[u_-(r)]^{-1}$ grows (as $r^{1/2}$) as $r\to\infty$, so

$$\lim_{r\to\infty}\frac{(2m)^{-1}\int_{r_1}^r dr'\,u_+(r')g(r')}{rg(r)[u_-(r)]^{-1}} = \lim_{r\to\infty}\frac{(2m)^{-1}u_+(r)g(r)}{rg(r)[u_-(r)]^{-1}\{r^{-1} + (\ln g)'(r) - (\ln u_-)'(r)\}}$$

$$= (2m)^{-1}\lim_{r\to\infty}[r\{\tfrac{1}{2}r^{-1} + O(r^{-2})\}]^{-1} = m^{-1}$$

because $(\ln g)'(r) = -2 - \frac{1}{2}r^{-1} + O(r^{-2})$ as $r\to\infty$; thus

$$u(r) = \alpha_- k_0(mr)\cdot(1 + o(\exp[-m(1-\epsilon)^{1/2}r])) + m^{-1}r^{1/2}[\beta' k_1(r)]^2\cdot(1 + o(1))$$

as $r\to\infty$.

Recycling these formulae through Lemma 3.2 shows that the factors $(1-\epsilon)^{1/2}$ may be eliminated. The same analysis gives the asymptotic behavior of u' and v'; we omit the details. Finally, application of the definitions of u and v in terms of R and S and of k_ν in terms of k_ν finishes the proof.

Let us make some remarks about the constants α and β appearing in Theorem 1. First of all, β must be nonzero because v cannot vanish identically; the sign of β may be determined as follows. Note that

$$\frac{d}{dr}(v(r)v'(r)) = [(1 + r^{-1/2}u(r))^2 + \tfrac{3}{4}r^{-2}]v^2(r) + [v'(r)]^2$$

so that $v\cdot v'$ is strictly increasing; but $\lim_{r\to\infty} v(r)v'(r) = 0$, so we see that $v(r)v'(r) < 0$ for all r. Since $(\mathrm{sgn}\,n)\cdot v(r)$ is positive for small enough r, [because[2] $S(r)\to 0$ as $r\to 0$] we conclude that $(\mathrm{sgn}\,n)\cdot v(r)$ is positive for all r, whence $(\mathrm{sgn}\,n)\cdot\beta$ is positive. It seems that we cannot argue in this way to show that α is positive, but under the assumption that $|R(r)|\leqslant 1$ for large r it is clear that $\alpha > 0$.

Using a different approach we may derive inequalities which α and β satisfy in certain circumstances. Define the function i_ν by

$$i_\nu(r) = (2\pi r)^{1/2} I_\nu(r),$$

where I_ν is the usual[4] modified Bessel function of order ν that is subdominant at the origin; i_ν satisfies the same differential equation as does k_ν, and $i_\nu' k_\nu - i_\nu k_\nu' = 2$.

If we write the equation satisfied by v in the form

$$-v''(r) + [1 + \tfrac{3}{4}r^{-2}]v(r) = (1 - R^2(r))\cdot r^{-1/2}(n + S(r)) = f_v(r)$$

we find that $[i_1' v - i_1 v']' = i_1 f_v$. By the proofs of Lemma 3.2 and Theorem 1,

$$i_1'(r)v(r) - i_1(r)v'(r) = 2\beta' + o(1) = (2\pi)^{1/2}\beta + o(1)$$

as $r\to\infty$. On the other hand, one may deduce[2] from the behavior of i_1 and v as $r\to 0$ that

$$i_1'(r)v(r) - i_1(r)v'(r) = (2\pi)^{1/2}n + o(1)$$

as $r\to 0$. Therefore

$$(2\pi)^{1/2}(\beta - n) = \int_0^\infty dr\, i_1(r) f_v(r).$$

Since $(\mathrm{sgn}\,n)\cdot f_v > 0$ if $|R|\leqslant 1$ (and $n\neq 0$) we find in particular that

$$(\mathrm{sgn}\,n)\cdot\beta > |n|$$

under the assumption that $|R|\leqslant 1$.

In a similar fashion we may write

$$-u''(r) + [m^2 - \tfrac{1}{4}r^{-2}]u(r) = r^{1/2}[V'(R(r)) + V''(1)(1 - R(r)) + r^{-2}(S(r) + n)^2 R(r)] = f_u(r)$$

and show that[2]

$$(2\pi m)^{1/2}\alpha = 2m\alpha' = [m i_0'(mr)u(r) - i_0(mr)u'(r)]_0^\infty = \int_0^\infty dr\, i_0(mr) f_u(r),$$

so long as $m < 2$. Therefore

$$\alpha > 0$$

in case $R\geqslant 0$ and the potential V satisfies

$$V'(\hat{R}) + V''(1)(1 - \hat{R})\geqslant 0$$

for all $\hat{R}\geqslant 0$. For example, the quartic double-well potential V_m defined by

$$V_m(\hat{R}) = \tfrac{1}{8}m^2(1 - \hat{R}^2)^2$$

satisfies this condition. We note that for the vortex solutions that have been constructed[2] the field R satisfies $0\leqslant R\leqslant 1$.

4. THE YANG–MILLS HIGGS MODEL

The Yang–Mills Higgs model describes a scalar Higgs field which is self-coupled via a potential V and which interacts with a Yang–Mills [i.e., SU(2)] gauge field in three-dimensional Euclidean space. The Higgs field transforms according to some finite dimensional, real, symmetric representation $g\mapsto U(g)$: $W\to W$ of the Lie group SU(2) in the vector space W, and thus ϕ is a W-valued function on $\mathbb{R}^3$; the gauge field is a 1-form on $\mathbb{R}^3$ taking values in the Lie algebra su(2) of SU(2). The potential V is assumed to be twice continuously differentiable, nonnegative, and symmetric about the origin, and to have a zero at $R_\infty > 0$ with $V''(R_\infty) > 0$ and $V(\hat{R}) > 0$ for $\hat{R} < R_\infty$. Using units in which $R_\infty = 1$ and $e/\hbar c = 1$ (where e is the coupling constant for the interaction between the Higgs and gauge fields) the Euclidean action for the Yang-Mills Higgs model is

$$\mathscr{A}(\phi, A) = \int_{\mathbb{R}^3} d^3x\,\{\tfrac{1}{2}\|F(A)\|^2 + \tfrac{1}{2}\|d_A\phi\|^2 + V(\|\phi\|)\}, \quad (4.1)$$

where $F(A) = dA + \frac{1}{2}[A, A]$ is the field strength (curvature) of A, and $d_A\phi = d\phi + U(A)\phi$ is the covariant derivative of ϕ. The criticial points of $\mathscr{A}$ formally satisfy the equations

$$d_A^* d_A\phi + V''(\|\phi\|)\phi/\|\phi\| = 0 \quad (4.2a)$$

and

$$d_A^* F(A) + J_A(\phi) = 0; \quad (4.2b)$$

here the Higgs current $J_A(\phi)$ is defined so that

$$(X|J_A(\phi))_{\mathrm{su}(2)} = (U(X)\phi\,|d_A\phi)_W$$

for all $X\in su(2)$. Monopoles are solutions of Eqs.(4.2a) and (4.2b) which have finite action (4.1) and exhibit symmetry breaking ($\|\phi(x)\|\to 1$ as $|x|\to\infty$).

One may construct[2,5–7] monopoles which are isotropic in the sense that

$$\phi(x) = R(|x|)\sum_m \tau^m Y^l_m\left(\frac{x}{|x|}\right) \quad (4.3a)$$

and

$$A(x) = S(|x|)\sum_{a,j,k}\frac{\sigma^a}{2i}\epsilon_{ajk}\frac{x^j dx^k}{|x|^2} \quad (4.3b)$$

for some integer l, where the σ^a, $a = 1, 2, 3$, are the usual

Pauli spin matrices, and the $\tau^m, m = -l,...,l$, transform according to the l th irreducible representation of SU(2) in such a way that

$$(1/i)L^a\phi + U(\sigma^a/2i)\phi = 0$$

(here $L^a = \Sigma_{j,k}\epsilon^{ajk}x_j(1/i)\,\partial/\partial x^k$ is the usual angular momentum operator). For example, we may take

$$\phi(x) = R(|x|)\sum_a \frac{\sigma^a}{2i}\frac{x^a}{|x|}$$

if ϕ transforms according to the $l = 1$ adjoint representation, as first shown by t'Hooft[8] and Polyakov.[9]

Assuming a suitable normalization for τ^m, the fields R and S may be shown to satisfy the differential equations

$$-R''(r) - 2r^{-1}R'(r) + l(l+1)r^{-2} \times (S(r)+1)^2R(r) + V'(R(r)) = 0 \tag{4.4a}$$

and

$$-S''(r) + \tfrac{1}{2}l(l+1)R^2(r)(S(r)+1) + r^{-2}S(r)(S(r)+1)(S(r)+2) = 0, \tag{4.4b}$$

to have

$$F(R,S) = \int_0^\infty r^2\,dr\,\{\tfrac{1}{2}[R'(r)]^2 + r^{-2}[S'(r)]^2 + \tfrac{1}{2}l(l+1)(S(r)+1)^2R^2(r) + \tfrac{1}{2}r^{-4}S^2(r)(S(r)+2)^2 + V(R(r))\}, \tag{4.5}$$

finite, and to satisfy

$$\lim_{r\to\infty} R(r) = 1. \tag{4.6}$$

Theorem 2: *Suppose that* R and S satisfy (4.4)–(4.6); let $m = [V''(1)]^{1/2}, m_l = [\tfrac{1}{2}l(l+1)]^{1/2}$, *and* $v_0 = i\,3^{1/2}/2$. Then there exist constants α and β such that

$$S(r) = -1 + \beta(m_lr)^{1/2}K_{v_0}(m_lr) \times (1 + o(\exp[-\min\{m,2m_l\}r]))$$

and

$$S'(r) = -m_l(S(r)+1)\cdot(1 + O(r^{-2}))$$

as $r\to\infty$, and such that:

(a) if $m < 2m_l$,

$$R(r) = 1 - \alpha(mr)^{-1}e^{-mr}(1 + o(\exp[-mr])) - 2m_l^4[m^2 - 4m_l^2]^{-1}\beta^2(m_lr)^{-1}[K_{v_0}(m_lr)]^2\cdot(1 + o(1))$$

and

$$R'(r) = -m(R(r)-1)\cdot(1 + (mr)^{-1} + O(r^{-2}));$$

(b) if $m = 2m_l$

$$R(r) = 1 - \tfrac{1}{2}m_l^2\beta^2\ln(m_lr)[K_{v_0}(m_lr)]^2\cdot(1 + o(1))$$

and

$$R'(r) = -m_l(R(r)-1)\cdot(1 + o(1));$$

(c) if $m > 2m_l$,

$$R(r) = 1 - 2m_l^4[m^2 - 4m_l^2]^{-1}\beta^2(m_lr)^{-1} \times [K_{v_0}(m_lr)]^2\cdot(1 + o(1))$$

and

$$R'(r) = -m_l(R(r)-1)\cdot(1 + o(1))$$

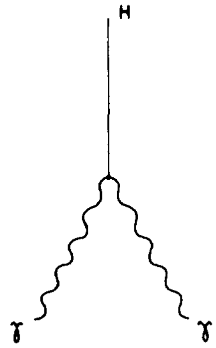

FIG. 2. The Feynman vertex coupling a Higgs particle to two massive photons.

as $r\to\infty$.

The proof[2] of Theorem 2 is exactly analogous to the proof of Theorem 1, so we spare the reader from the details. As far as the constants α and β are concerned, clearly $\beta \neq 0$, and $\beta > 0$ if $S(r) \geqslant -1$ for all large r; equally clearly $\alpha > 0$ if $|R(r)| \leqslant 1$ for all large r, and[2] one may show, in the manner of Sec. 3, that $\alpha > 0$ in case $R \geqslant 0$ and the potential satisfies the inequality (3.3).

The asymptotic behavior of physical fields such as $\|\phi\| = |R|, (\phi\,|d_A\phi)_W = R\cdot R'\,dr$, $\|\phi\|^2\|d_A\phi\|^2 - \|(\phi\,|d_A\phi)_W\|^2 = l(l+1)R^4(S+1)^2$, $\|F(A)\|^2 = 2r^{-2}(S')^2 + r^{-4}S^2(S+2)^2$, and (in case $l = 1$) $(\phi\,|F(A))_{su(2)} = R\cdot S(S+2)d\theta\wedge\sin\theta\,d\phi$. In particular, the mass of the gauge particle $m_A = \hbar/c\cdot e/\hbar c\cdot R_\infty\cdot[\tfrac{1}{2}l(l+1)]^{1/2}$, while the mass of the Higgs boson is $m_\phi = \min\{\hbar/c\cdot[V''(R_\infty)]^{1/2}, 2m_A\}$.

5. CONCLUSION

Let us comment on the physics behind the result that the mass of the Higgs field cannot exceed twice the mass of the gauge field. In the case of the abelian Higgs model, it is seen from the proofs of Theorem 1 that this arises because the differential equation (3.2a) for the shifted field u has an inhomogeneous term $r^{-1/2}v^2(r)$ whose decay is twice that of the gauge field. In the context of the quantized version of this model, the term in the Euclidean action which gives rise to this inhomogeneous term in the field equations corresponds to the Fenynman vertex shown in Fig. 2 which describes the decay of a Higgs particle into two massive photons. Thus the peculiarity in the Higgs particle mass in the classical field theory reflects the existence of a decay mode $H\to 2\gamma$ in the quantum field theory. A similar interpretation is possible for the Yang-Mills Higgs model.

Finally, we note that Jaffe and Taubes[10] have studied nonisotropic vortices and monopoles with the quartic double-well potential $V_{mc/\hbar}$ and have established that $m_\phi \geqslant \min\{m,2m_A\}$; they conjecture that $m_\phi = \min\{m,2m_A\}$ in this more general setting.

ACKNOWLEDGMENTS

The author thanks Erhard Seiler for suggesting this problem and for his patient guidance during the course of this work, and Mark Ashbaugh for many interesting discussions.

[1]H. Nielsen and P. Olesen, "Vortex Models for Dual Strings," Nucl. Phys. B **61**, 361 (1973).
[2]B. Plohr, Princeton University Doctoral Dissertation, October 1980.
[3]M. Protter and H. Weinberg, *Maximum Principles in Differential Equations* (Prentice-Hall, Englewood Cliffs, N. J., 1967).
[4]Bateman Manuscript Project, *Higher Transcendental Functions*, Vol. II (McGraw-Hill, New York, 1953).
[5]Yu. Tyupkin, V. Fateev, and A. Svarts, "Particle-like Solutions of the Equations of Gauge Theories," Theoret. and Math. Phys. **26**, 270 (1977).
[6]J. Rawnsley, Dublin Institute for Advanced Studies Preprint DIAS-TP-77-09, 1977; "Spherically Symmetric Monopoles are Smooth," J. Phys. A **10**, L139 (1977).
[7]R. Weder, "Existence, Regularity, and Exponential Decay of Finite Energy Solutions to Gauge Fields," Communicaciones Técnicas **10**, 198 (1979).
[8]G. t'Hooft, "Magnetic Monopoles in Unified Gauge Theories," Nucl. Phys. B **79**, 276 (1974).
[9]A. Polyakov, "Particle Spectrum in Quantum Field Theory," ZhETF Pis. Red. **20**, 430 (1974) [JETP Lett. **20**, 194 (1974)].
[10]A. Jaffe and C. Taubes, *Vortices and Monopoles* (Birkhäuser, Boston, 1980).