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A Conservative Eulerian Formulation of the Equations for Elastic Flow*

BRADLEY J. PLOHR

Computer Sciences Department and Center for the Mathematical Sciences, University of Wisconsin-Madison, Madison, Wisconsin 53706

AND

DAVID H. SHARP

Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545

We show how the partial differential equations governing elasticity can be written in fully conservative, first-order, Eulerian form. This permits investigation of wave solutions of the equations of elasticity by a combination of mathematical analysis and high-quality numerical methods based on Riemann solutions of the governing equations.

1. INTRODUCTION

The partial differential equations describing the deformation of an elastic body support a rich variety of waves (shock, shear, and torsion) as solutions. A scientific understanding of these wave solutions requires a mathematical analysis of the structure of the governing equations, combined with numerical computation of the solutions.

When studying quasi-linear hyperbolic partial differential equations, whose solutions typically develop discontinuities, one must begin by writing the equations as a system of conservation laws. Assuming that there are n independent conserved quantities, the solution is specified by a vector $u(x, t)$ that lies in an n-dimensional state space. The dynamical equations for u take the form

$$\frac{\partial u}{\partial t} + \nabla \cdot f(u) = 0 \tag{1.1}$$

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of a first-order system of partial differential equations in divergence form; $f(u)$ is the vector of fluxes associated with the conserved quantities.

A Riemann problem is an initial-value problem for Eq. (1.1) with scale-invariant initial data. Riemann problems are of fundamental importance in a study of the wave phenomena associated with a system of conservation laws: they determine both the long-time asymptotic behavior of a general solution and the instantaneous interaction between pairs of nonlinear waves. Specializing further to scale-invariant solutions that propagate at a fixed speed defines elementary waves. Elementary waves are both more fundamental and more elementary than solutions of Riemann problems. They are the building blocks out of which solutions of Riemann problems are constructed.

Elementary waves and Riemann solutions form the bridge between the mathematical analysis of conservation laws and high-quality methods for numerical computation of their solutions. For instance, higher order Godunov methods [1] require a conservative form for the dynamical equations together with approximate solutions of Riemann problems, while front-tracking schemes [2] place an even greater reliance on Riemann solutions.

The analysis of conservation laws with Riemann problems can, to a great extent, be carried out in either the Lagrangian or the Eulerian picture. The comparative advantage of a Lagrangian vs. an Eulerian description of a flow is a problem-dependent matter. Our intention is to develop methods whose scope includes large deformations of the body, which may lead to plastic or other forms of structural failure. At the level of numerical computation, this disposes one to an Eulerian description since Lagrangian methods, traditionally used for solids when deformations are small, are subject to failure (e.g., by mesh tangling) at large deformations. By contrast, Eulerian methods, especially as enhanced by front tracking, can be made robust and accurate in such circumstances.

The synthesis of mathematical analysis and numerical computation along the lines mentioned above has found extremely fruitful application in studying wave solutions in compressible fluid dynamics. We wish to study wave solutions in elasticity and elastoplasticity from a similar standpoint. As a first step, we show in this paper how the partial differential equations governing elasticity can be written in fully conservative, first-order, Eulerian form.

To carry out this program, we begin with the familiar Eulerian form of the equations of mass, momentum, and energy conservation in the theory of elasticity. A first-order form of the dynamical equations is achieved by introducing an Eulerian measure of the deformation gradient and showing that its evolution can be formulated as a conservation law. The system of equations is completed with an equation of state expressing the energy as a function of the Eulerian deformation gradient and the entropy. This makes it unnecessary to use differential constitutive relations, which are not in conservation form, to propagate the stress and temperature.

In Section 2 we discuss the formulation of the equations governing elasticity in the Lagrangian picture. The discussion is divided into three parts: the kinematical description of deformable bodies and the conservation laws that govern them; constitutive relations that characterize the material; and the jump conditions that determine the structure of discontinuous waves. In Section 3 we translate the results in the Lagrangian picture to the Eulerian picture. This section also is divided into three parts, in parallel to Section 2. Finally in Section 4 we summarize the conservative Eulerian formulation of elasticity.

2. Elasticity in the Lagrangian Picture

In this section we discuss the Lagrangian (i.e., material) formulation of elasticity. It provides the basis for our derivation, in the next section, of the conservation form of elasticity in the Eulerian (i.e., spatial) picture.

2a. *Kinematics and Conservation Laws*

We first discuss the kinematics and conservation laws obeyed by deformable bodies in the Lagrangian picture. We refer the reader to the literature (e.g., Refs. [4, 5, 7]) for more details. The motion of a body $\mathscr{B}$ is represented mathematically by a time-dependent map ϕ embedding $\mathscr{B}$ into an ambient space $\mathscr{S}$. Let X^α, $\alpha = 1,2,3$ denote *material coordinates* on $\mathscr{B}$, and let x^i, $i = 1,2,3$ denote *spatial coordinates* on $\mathscr{S}$. Then

$$x^i = \phi^i(X,t); \tag{2.1}$$

ϕ is assumed to be piecewise smooth (i.e., continuously differentiable).

The motion ϕ is required to respect the conservation of mass, momentum, and energy. These principles are expressed as integral conditions involving the temporal and spatial derivatives of ϕ, namely the *Lagrangian velocity*

$$V^i = \frac{\partial \phi^i}{\partial t} \tag{2.2}$$

and the *Lagrangian deformation gradient*

$$F^i{}_\alpha = \frac{\partial \phi^i}{\partial X^\alpha}. \tag{2.3}$$

Let ρ_{ref} denote the *mass density* of the undeformed body, $\mathscr{E}$ the specific energy, $S^{\alpha\beta}$ the (second) *Piola–Kirchhoff stress tensor*, and Q^{α} the *Lagrangian heat flux*. Then the integral conservation laws take the form

$$\frac{d}{dt}\int_{\mathscr{U}} \rho_{\text{ref}}\, d\mathbf{V} = 0, \tag{2.4}$$

$$\frac{d}{dt}\int_{\mathscr{U}} \rho_{\text{ref}} V^i\, d\mathbf{V} = \oint_{\partial\mathscr{U}} F^i{}_{\alpha} S^{\alpha\beta} N_{\beta}\, d\mathbf{A}, \tag{2.5}$$

and

$$\frac{d}{dt}\int_{\mathscr{U}} \rho_{\text{ref}}\left(\tfrac{1}{2}V_i V^i + \mathscr{E}\right) d\mathbf{V} = \oint_{\partial\mathscr{U}} V_i F^i{}_{\alpha} S^{\alpha\beta} N_{\beta}\, d\mathbf{A} - \oint_{\partial\mathscr{U}} Q^{\gamma} N_{\gamma}\, d\mathbf{A} \tag{2.6}$$

for any region $\mathscr{U}$ in $\mathscr{B}$ with a piecewise smooth boundary $\partial\mathscr{U}$ and unit outward normal N_{α}. (For simplicity we omit body forces and heat sources.) Additionally, the conservation of angular momentum may be shown to lead to the requirement that

$$S^{\beta\alpha} = S^{\alpha\beta}. \tag{2.7}$$

When the fluid variables are smooth, the integral conservation laws are equivalent to corresponding partial differential equations:

$$\dot{\rho}_{\text{ref}} = 0, \tag{2.8}$$

$$\rho_{\text{ref}} \dot{V}^i = \left(F^i{}_{\alpha} S^{\alpha\beta}\right)_{;\beta}, \tag{2.9}$$

and

$$\rho_{\text{ref}}\left(\tfrac{1}{2}V_i V^i + \mathscr{E}\right)^{\boldsymbol{\cdot}} = \left(V_i F^i{}_{\alpha} S^{\alpha\beta}\right)_{;\beta} - Q^{\gamma}{}_{;\gamma}. \tag{2.10}$$

Here the dot and the semicolon denote differentiation with respect to time and space. (We adopt this notation for generality as well as conciseness: the formulae throughout the paper remain valid when curvilinear coordinates are used for the spaces $\mathscr{B}$ and $\mathscr{S}$ provided that the dot and semicolon are interpreted as covariant derivatives and that Kronecker delta symbols are replaced by metric tensors.)

Using Eq. (2.2), we see that the evolution of ϕ is governed by a system of conservation laws involving the second time derivative of ϕ. The state of the material is characterized by the three components of the deformation, the three components of the velocity, and an additional variable to determine the local thermodynamic state (e.g., the entropy). If the energy, stress, and heat flux are known, the evolution of these seven degrees of freedom is determined by conservation of momentum and energy.

The deformation ϕ enters the conservation laws only through its derivatives, so the second-order equations may be regarded as first-order equations in which the velocity and the deformation gradient are taken to be fundamental dynamical variables. If the three components of ϕ are eliminated in favor of the nine components of $F^i{}_\alpha$, however, the system of equations must be supplemented to accommodate the expanded set of variables. The additional equations may also be written as integral conservation laws:

$$\frac{d}{dt}\int_{\mathscr{U}} F^i{}_\alpha \, d\mathbf{V} = \oint_{\partial \mathscr{U}} V^i N_\alpha \, d\mathbf{A}, \tag{2.11}$$

as follows from Gauss' law. (See Ref. [3] for more discussion.) When V^i and $F^i{}_\alpha$ are smooth, this integral conservation law is also equivalent to a set of differential equations

$$\dot{F}^i{}_\alpha = V^i{}_{;\alpha}. \tag{2.12}$$

Six of these equations act as constraints on the deformation gradient,

$$\left(F^i{}_{\alpha;\beta} - F^i{}_{\beta;\alpha}\right)^{\cdot} = 0, \tag{2.13}$$

so that $F^i{}_\alpha$ is a gradient for all time, provided that it is a gradient at some initial time. As will be discussed shortly, the remaining three equations contain bona fide dynamical information.

For later purposes we introduce the *deformation rate* and *vorticity tensors*

$$D_{\alpha\beta} = \tfrac{1}{2}\left(F^k{}_\alpha V_{k;\beta} + V_{k;\alpha} F^k{}_\beta\right), \tag{2.14}$$

$$\Omega_{\alpha\beta} = \tfrac{1}{2}\left(F^k{}_\alpha V_{k;\beta} - V_{k;\alpha} F^k{}_\beta\right). \tag{2.15}$$

We also define the *Lagrangian strain tensor*

$$E_{\alpha\beta} = \tfrac{1}{2}\left(F_{k\alpha} F^k{}_\beta - \delta_{\alpha\beta}\right). \tag{2.16}$$

Notice that

$$\dot{E}_{\alpha\beta} = D_{\alpha\beta} \tag{2.17}$$

by virtue of Eq. (2.12).

2b. *Constitutive Relations*

The conservation laws, as formulated in Eqs. (2.8)–(2.10) and (2.12), involve the stress, energy, and heat flux. These equations are incomplete unless supplemented by a suitable set of constitutive relations, which may take various forms. Frequently they specify the dynamical evolution of the stress and temperature. This is natural in computational methods for solving the evolution equations: the values of the fluid variables are incremented over successive time steps, so the increments need to be determined from the current values. However, this approach involves equations that are not conservation laws. We will show in this section how the constitutive relations can be formulated in a manner that is compatible with a conservative form for the dynamical equations.

In this paper we will write the constitutive relation for the heat flux in the simple form

$$Q^{\alpha} = -\mathscr{K} T^{;\alpha}, \tag{2.18}$$

which expresses the heat flux in terms of the temperature T and the *thermal conductivity* $\mathscr{K}$. It remains to specify the stress and the temperature. For elastic media the dynamical equations for the stress and temperature take the form

$$\dot{S}^{\alpha\beta} = \mathbf{C}^{\alpha\beta\gamma\delta} D_{\gamma\delta} + \Gamma^{\alpha\beta} Q^{\gamma}{}_{;\gamma} \tag{2.19}$$

and

$$\dot{T} = -T\Gamma^{\alpha\beta} D_{\alpha\beta} - \frac{1}{\rho_{\text{ref}} C_{\eta}} Q^{\gamma}{}_{;\gamma}, \tag{2.20}$$

where the $\mathbf{C}^{\alpha\beta\gamma\delta}$ are the *adiabatic elastic moduli*, the $\Gamma^{\alpha\beta}$ are the *Grüneisen coefficients*, and C_{η} is the *specific heat at constant elastic configuration*. These coefficients are required to satisfy symmetry properties (which follow from thermodynamic consistency when the coefficients are derived from an equation of state; see below):

$$\mathbf{C}^{\beta\alpha\gamma\delta} = \mathbf{C}^{\alpha\beta\delta\gamma} = \mathbf{C}^{\gamma\delta\alpha\beta} = \mathbf{C}^{\alpha\beta\gamma\delta} \tag{2.21}$$

and

$$\Gamma^{\beta\alpha} = \Gamma^{\alpha\beta}. \tag{2.22}$$

If the coefficients are known as functions of the fluid variables, then Eqs. (2.8)–(2.10) and (2.12), together with Eqs. (2.18)–(2.20), form a complete set of evolution equations. However, this formulation suffers from two deficiencies. First, Eqs. (2.19)–(2.20) are not in conservation form. Second, the coefficients $\mathbf{C}^{\alpha\beta\gamma\delta}$, $\Gamma^{\alpha\beta}$, and C_{η} cannot be specified arbitrarily; rather they must be compatible with the principles of thermodynamics. We will see that addressing these problems leads to a more concise expression of the information contained in the coefficients.

To reformulate the constitutive relations, we note that the stress and temperature can be derived from an equation of state characterizing the material [7]. For elastic materials, the specific energy $\mathscr{E}$ is given as a function of the deformation gradient $F^i{}_\alpha$ and the specific entropy S. Because of the invariance of $\mathscr{E}$ with respect to rigid-body transformations of the spatial coordinates (the principle of frame indifference), $\mathscr{E}$ depends on $F^i{}_\alpha$ only through the Lagrangian strain tensor $E_{\alpha\beta}$. Thus we postulate the equation of state

$$\mathscr{E} = U(E_{\alpha\beta}, S). \tag{2.23}$$

(By writing U without explicit dependence on the material position X we are assuming that the material is *homogeneous*. To be consistent, we assume hereafter that the Lagrangian density ρ_{ref} is constant throughout $\mathscr{B}$.)

To identify suitable derivatives of the energy with the stress and temperature, we invoke a thermodynamic principle, the Clausius–Duhem inequality [7]

$$\rho_{\text{ref}}\dot{S} + (Q^\alpha/T)_{;\alpha} \geq 0. \tag{2.24}$$

First notice that the conservation of energy, Eq. (2.10), may be combined with Eqs. (2.8)–(2.9) to obtain the equivalent equation

$$\rho_{\text{ref}}\dot{\mathscr{E}} = S^{\alpha\beta}D_{\alpha\beta} - Q^\gamma{}_{;\gamma}. \tag{2.25}$$

Now the Clausius–Duhem inequality, when combined with Eqs. (2.25), (2.17), and the equation of state, shows that

$$\left(S^{\alpha\beta} - \rho_{\text{ref}}\frac{\partial U}{\partial E_{\alpha\beta}}\right)\dot{E}_{\alpha\beta} + \rho_{\text{ref}}\left(T - \frac{\partial U}{\partial S}\right)\dot{S} - \frac{Q^\gamma T_{;\gamma}}{T} \geq 0. \tag{2.26}$$

Assuming that $\dot{E}_{\alpha\beta}$, $\dot{S}$, and $T_{;\gamma}$ can be chosen arbitrarily leads to the identifications

$$S^{\alpha\beta} = \rho_{\text{ref}}\frac{\partial U}{\partial E_{\alpha\beta}} \tag{2.27}$$

and

$$T = \frac{\partial U}{\partial S}, \tag{2.28}$$

as well as the constraint

$$-Q^\gamma T_{;\gamma} \geq 0 \tag{2.29}$$

and the entropy equation

$$\rho_{\text{ref}}T\dot{S} = -Q^\gamma{}_{;\gamma}. \tag{2.30}$$

The evolution equations (2.19)–(2.20) for stress and temperature now follow from the above by straightforward computations, provided that the coefficients $\mathbf{C}^{\alpha\beta\gamma\delta}$, $\Gamma^{\alpha\beta}$, and C_η are identified as second derivatives of the equation of state:

$$\mathbf{C}^{\alpha\beta\gamma\delta} = \rho_{\text{ref}} \frac{\partial^2 U}{\partial E_{\alpha\beta}\, \partial E_{\gamma\delta}}, \tag{2.31}$$

$$-T\Gamma^{\alpha\beta} = \frac{\partial^2 U}{\partial E_{\alpha\beta}\, \partial S}, \tag{2.32}$$

and

$$\frac{T}{C_\eta} = \frac{\partial^2 U}{\partial S^2}. \tag{2.33}$$

The symmetry conditions (2.21)–(2.22) are automatically satisfied.

With the definitions (2.23), (2.27)–(2.28), and (2.18), Eqs. (2.8)–(2.10) and (2.12) comprise a complete system of evolution equations in conservation form that describes elastic materials. There are 14 equations for the 14 fluid variables ρ_{ref}, $F^i{}_\alpha$, V^i, and S. Since 6 of these equations constrain F^i_α to be a gradient, there are 8 physical degrees of freedom, which correspond to 8 wave modes.

The density and entropy mode propagate at zero speed, as seen from Eqs. (2.8) and (2.30). To determine the other modes, notice that Eq. (2.9) may be written

$$\rho_{\text{ref}} \dot{V}^i = \mathbf{A}^{i\beta j\delta} F_{j\delta;\beta} - \rho_{\text{ref}} T F^i{}_\alpha \Gamma^{\alpha\beta} S_{;\beta}, \tag{2.34}$$

where

$$\mathbf{A}^{i\beta j\delta} = \rho_{\text{ref}} \frac{\partial^2 U}{\partial F_{i\beta}\, \partial F_{j\delta}} = F^i{}_\alpha \mathbf{C}^{\alpha\beta\gamma\delta} F^j{}_\gamma + \delta^{ij} S^{\beta\delta} \tag{2.35}$$

is the *elasticity tensor*; this follows because $F^i{}_\alpha S^{\alpha\beta} = \rho_{\text{ref}} \partial U / \partial F_{i\beta}$. Consequently, a plane wave with normal N_α must propagate with speed S_N such that $\rho_{\text{ref}} S_N^2$ is an eigenvalue of the *acoustic tensor*

$$\mathbf{A}^{ij}_N = N_\beta \mathbf{A}^{i\beta j\delta} N_\delta. \tag{2.36}$$

Assuming the eigenvalues of $\mathbf{A}^{ij}_N$ to be positive, the wave modes occur in three pairs propagating with equal and opposite velocity. For isotropic materials, the fastest wave corresponds to longitudinal stress (i.e., pressure) modes, while the two slower waves correspond to radial shear stress and angular shear stress (i.e., torsion) modes.

2c. *Jump Conditions*

When the fluid variables are not smooth the integral forms of the conservation laws remain valid, but their formulations as partial differential equations hold only in the weak sense. In particular, if the variables are piecewise smooth with a jump discontinuity across a surface in the body $\mathscr{B}$, the solution satisfies the Rankine–Hugoniot jump conditions, which follow from the transport theorem for discontinuous solutions.

To express these conditions, let the discontinuity surface locally divide space-time into two regions, $\mathscr{U}_-$ and $\mathscr{U}_+$, and let the unit normal point from $\mathscr{U}_-$ to $\mathscr{U}_+$. Denote the jump in a quantity A across the discontinuity by $\Delta A = A_+ - A_-$, and the average by $\overline{A} = \frac{1}{2}(A_+ + A_-)$. Also, let N_α denote the unit normal and S_N the normal propagation speed for the discontinuity surface. Then the jump condition corresponding to conservation of mass, Eq. (2.4), is

$$-S_N \Delta\rho_{\text{ref}} = 0, \tag{2.37}$$

which shows that the *Lagrangian mass flux* $M = \rho_{\text{ref}} S_N$ suffers no jump discontinuity across the front. Therefore the jump conditions corresponding to the conservation laws (2.5)–(2.6) and (2.11) may be written

$$-M\Delta V^i = \Delta\left(F^i{}_\alpha S^{\alpha\beta}\right) N_\beta, \tag{2.38}$$

$$-M\Delta\left(\tfrac{1}{2}V_i V^i + \mathscr{E}\right) = \Delta\left(V_i F^i{}_\alpha S^{\alpha\beta}\right) N_\beta - \Delta Q^\gamma N_\gamma, \tag{2.39}$$

and

$$-M\Delta\left(\rho_{\text{ref}}^{-1} F^i{}_\alpha\right) = \Delta V^i N_\alpha. \tag{2.40}$$

3. Elasticity in the Eulerian Picture

The conservation laws (2.4)–(2.6) and (2.11) are formulated in the Lagrangian picture. In this section we will translate them into the Eulerian picture.

3a. *Kinematics and Conservation Laws*

We introduce the *Jacobian determinant* J of the map ϕ, which is the determinant of the linear transformation $F^i{}_\alpha$. The Jacobian is involved in two fundamental identities: the change of variables for integrals and the Piola identity [5]. Formally, the change of variables formula introduces the factor J^{-1} when an integral over $\mathscr{U}$ is transformed to an integral over $\mathscr{U}_t = \phi_t[\mathscr{U}]$, while the Piola identity replaces N_α by $J^{-1} n_i F^i{}_\alpha$ when an integral over $\partial\mathscr{U}$ is transformed to an integral over $\partial\mathscr{U}_t$, which has unit outward normal n_i. These factors account for the change in volume and area caused by ϕ: $d\mathbf{v} = J\,d\mathbf{V}$ and $n_i\,d\mathbf{a} = JN_\alpha(F^{-1})^\alpha{}_i\,d\mathbf{A}$. The differential form of the Piola identity is

$$\left\{J(F^{-1})^\alpha{}_i\right\}_{;\alpha} = 0, \tag{3.1}$$

as follows from the calculation

$$\int_{\mathscr{U}}\left\{J(F^{-1})^\alpha{}_i\right\}_{;\alpha} d\mathbf{V} = \oint_{\partial\mathscr{U}} J(F^{-1})^\alpha{}_i N_\alpha\,d\mathbf{A} = \oint_{\partial\mathscr{U}_t} n_i\,d\mathbf{a} = 0. \tag{3.2}$$

Equation (3.1) may be verified directly using Cramer's formula for the inverse of a matrix.

To formulate the conservation laws in the Eulerian picture, we require notation for several Eulerian quantities:

$$\rho = J^{-1}\rho_{\text{ref}}, \tag{3.3}$$

$$\varepsilon = \mathscr{E}, \tag{3.4}$$

$$v^i = V^i, \tag{3.5}$$

$$\sigma^{ij} = J^{-1}F^i{}_\alpha S^{\alpha\beta}F^j{}_\beta, \tag{3.6}$$

$$q^i = J^{-1}F^i{}_\alpha Q^\alpha, \tag{3.7}$$

where the quantities on the left are regarded as functions of $x = \phi(X, t)$ and t. The tensor σ^{ij} is the *Cauchy stress tensor*. Then the change of variable formula and the Piola identity may be used in the standard fashion to show that the Lagrangian conservation laws (2.4)–(2.6) are equivalent to

$$\frac{d}{dt}\int_{\mathscr{U}_t}\rho\,d\mathbf{v} = 0, \tag{3.8}$$

$$\frac{d}{dt}\int_{\mathscr{U}_t}\rho v^i\,d\mathbf{v} = \oint_{\partial\mathscr{U}_t}\sigma^{ij}n_j\,d\mathbf{a}, \tag{3.9}$$

and

$$\frac{d}{dt}\int_{\mathscr{U}_t}\rho\left(\tfrac{1}{2}v_iv^i + \varepsilon\right)d\mathbf{v} = \oint_{\partial\mathscr{U}_t}v_i\sigma^{ij}n_j\,d\mathbf{a} - \oint_{\partial\mathscr{U}_t}q^kn_k\,d\mathbf{a}. \tag{3.10}$$

Conservation of angular momentum, Eq. (2.7), becomes

$$\sigma^{ji} = \sigma^{ij}. \tag{3.11}$$

Using the transport theorem (see, e.g. [5]) to calculate the left-hand sides in these equations, the Eulerian conservation laws become the partial differential equations

$$\frac{\partial \rho}{\partial t} + \left(\rho v^i\right)_{;i} = 0, \tag{3.12}$$

$$\frac{\partial}{\partial t}\left(\rho v^i\right) + \left(\rho v^i v^j\right)_{;j} = \sigma^{ij}{}_{;j}, \tag{3.13}$$

and

$$\frac{\partial}{\partial t}\left[\rho\left(\tfrac{1}{2}v_i v^i + \varepsilon\right)\right] + \left[\rho\left(\tfrac{1}{2}v_i v^i + \varepsilon\right)v^j\right]_{;j} = \left(v_i\sigma^{ij}\right)_{;j} - q^k{}_{;k} \tag{3.14}$$

(again assuming the fluid variables to be smooth). Another useful form of this system of equations is

$$\rho\left(\rho^{-1}\right)^{\cdot} = v^k{}_{;k}, \tag{3.15}$$

$$\rho\dot{v}^i = \sigma^{ij}{}_{;j}, \tag{3.16}$$

$$\rho\dot{\varepsilon} = \sigma^{ij}d_{ij} - q^k{}_{;k}. \tag{3.17}$$

Here the dot denotes the convective derivative

$$(\ )^{\cdot} = \frac{\partial}{\partial t}(\ ) + (\ )_{;i}v^i$$

when applied to an Eulerian tensor, and the *Eulerian deformation rate and vorticity tensors* are

$$d_{ij} = \tfrac{1}{2}\left(v_{i;j} + v_{j;i}\right) = \left(F^{-1}\right)^{\alpha}{}_{i}D_{\alpha\beta}\left(F^{-1}\right)^{\beta}{}_{j}, \tag{3.18}$$

$$\omega_{ij} = \tfrac{1}{2}\left(v_{i;j} - v_{j;i}\right) = \left(F^{-1}\right)^{\alpha}{}_{i}\Omega_{\alpha\beta}\left(F^{-1}\right)^{\beta}{}_{j}. \tag{3.19}$$

The main point of this section is that the conservation law (2.11) may be transformed to the Eulerian picture in the same way. If we introduce the *Eulerian deformation gradient*

$$f^i{}_{\alpha} = \rho_{\text{ref}}^{-1}F^i{}_{\alpha}, \tag{3.20}$$

which is regarded also as a function of $x = \phi(X, t)$ and t, then Eq. (2.11) becomes

$$\frac{d}{dt}\int_{\mathscr{U}_t}\rho f^i{}_{\alpha}\,d\mathrm{v} = \oint_{\partial\mathscr{U}_t}\rho v^i f^j{}_{\alpha}n_j\,d\mathbf{a}. \tag{3.21}$$

In differential form, this equation is

$$\frac{\partial}{\partial t}\left(\rho f^i{}_\alpha\right) + \left(\rho f^i{}_\alpha v^j\right)_{;j} = \left(\rho v^i f^j{}_\alpha\right)_{;j}, \tag{3.22}$$

Alternatively, this equation may be obtained directly from Eq. (2.12) by using conservation of mass (Eqs. (2.8) and (3.12)) and the Piola identity (Eq. (3.1)) as

$$\rho\left(\rho_{\text{ref}}^{-1} F^i{}_\alpha\right)^{\cdot} = J^{-1}\left(V^i F^j{}_\alpha (F^{-1})^\beta{}_j\right)_{;\beta} = \left(J^{-1} V^i F^j{}_\alpha\right)_{;\beta}(F^{-1})^\beta{}_j \tag{3.23}$$

so that

$$\rho \dot{f}^i{}_\alpha = \left(\rho v^i f^j{}_\alpha\right)_{;j}; \tag{3.24}$$

combined with Eq. (3.12), this yields Eq. (3.22). Equations (3.12)–(3.14) and (3.22) comprise the Eulerian system of conservation laws for elasticity that correspond to the Lagrangian form, Eqs. (2.8)–(2.10) and (2.12).

To establish the equivalence between Lagrangian and Eulerian pictures, we must demonstrate that the Lagrangian formulation follows from the Eulerian one. The passage from the Lagrangian picture to the Eulerian picture relies on the interpretation of $f^i{}_\alpha$ as the gradient of a deformation; in particular, the Piola identity is invoked. We now show that the Piola identity is contained in Eq. (3.22), in the following sense. Differentiating Eq. (3.22) with respect to x^i shows that

$$\frac{\partial}{\partial t}\left[\left(\rho f^i{}_\alpha\right)_{;i}\right] = 0. \tag{3.25}$$

Therefore if

$$\left(\rho f^i{}_\alpha\right)_{;i} = 0 \tag{3.26}$$

holds at any time $t = t_0$, it holds for all time. When ρ and $f^i{}_\alpha$ are related to ρ_{ref} and $F^i{}_\alpha$ as usual, Eq. (3.26) is equivalent to the Piola identity, Eq. (3.1), for the inverse of the deformation ϕ.

Using this result, we may show that the differential conservation laws in the Lagrangian picture, Eqs. (2.8)–(2.10) and (2.12), follow from the conservation laws in the Eulerian picture, Eqs. (3.12)–(3.14) and (3.22), provided that the Piola identity (3.26) holds at some initial time $t = t_0$. This assumes that the solution is smooth, so that the differential equations are meaningful; an analogous result for discontinuous solutions is demonstrated in Section 3c.

THEOREM 1. *Consider a smooth solution of the Eulerian conservation laws, Eqs.* (3.12)–(3.14) *and* (3.22). *Suppose that there is an initial deformation* ϕ_0 *and a constant* ρ_{ref} *such that at the initial time* $t = t_0$, $f^i{}_\alpha|_{t=t_0} = \rho_{\text{ret}}^{-1}\, \partial\phi_0^i/\partial X^\alpha$, *and* $\rho|_{t=t_0} = J_0^{-1}\rho_{\text{ref}}$, *where* J_0 *is the Jacobian of* ϕ_0. *(The quantities on the left are regarded as functions of* $x = \phi_0(X)$.) *Then there is a motion* ϕ *satisfying the Lagrangian conservation laws, Eqs.* (2.8)–(2.10) *and* (2.12), *and the initial condition* $\phi|_{t=t_0} = \phi_0$, *with the Lagrangian and Eulerian fluid variables related by Eqs.* (3.3)–(3.7) *and* (3.20).

Proof. As just argued, the Piola identity in the form Eq. (3.26) follows from Eq. (3.22) and the initial conditions. The identity and the conservation of mass equation may be combined with Eq. (3.22) to yield

$$\dot{f}^i{}_\alpha = v^i{}_{;j} f^j{}_\alpha. \tag{3.27}$$

Now define ϕ by solving the initial-value problem $\partial\phi^i/\partial t = v^i \circ \phi$, $\phi^i|_{t=t_0} = \phi_0^i$. Then Eq. (2.12) shows that $F^i{}_\alpha = \partial\phi^i/\partial X^\alpha$, regarded as a function of $x = \phi(X, t)$ and t, also satisfies Eq. (3.27):

$$\dot{f}^i{}_\alpha \left(f^{-1}\right)^\alpha{}_j = v^i{}_{;j} = \dot{F}^i{}_\alpha \left(F^{-1}\right)^\alpha{}_j. \tag{3.28}$$

Therefore $f^i{}_\alpha$ and $F^i{}_\alpha$ differ by a factor that is constant in time; according to the initial conditions, this factor is the same as in Eq. (3.20). Finally, the mass, momentum, and energy equations in the Lagrangian picture follow from the Eulerian equations by the standard manipulations. □

This theorem shows that the Eulerian formulation is independent of the identification of $f^i{}_\alpha$ as a gradient. Thus the Eulerian conservation laws may be solved without constructing the deformation ϕ or integrating along particle paths: no reference to Lagrangian quantities is needed. The structure of the Eulerian equations guarantees that the fluid variables describe the deformation of a material. We emphasize, however, that the conservation of mass equation must be regarded as separate from the conservation law for the Eulerian deformation gradient in order to recover the Lagrangian formulation.

3b. *Constitutive Relations*

The conservation laws in the Eulerian picture must also be supplemented by constitutive relations. The equations corresponding to Eqs. (2.18)–(2.20) in the Eulerian picture are obtained readily from the definitions and Eq. (2.12):

$$q^i = -\kappa\theta^{;i}, \tag{3.29}$$

$$\dot{\sigma}^{ij} + \sigma^{ik}\omega_k{}^j - \omega^i{}_k\sigma^{kj} = b^{ijkl} d_{kl} + \gamma^{ij} q^k{}_{;k}, \tag{3.30}$$

and

$$\dot{\theta} = -\theta\gamma^{ij}d_{ij} - \frac{1}{\rho c_\eta} q^k{}_{;k}. \tag{3.31}$$

Here the Eulerian quantities are defined by

$$\theta = T, \tag{3.32}$$

$$\eta = S, \tag{3.33}$$

$$\kappa = J^{-1}\mathscr{K}, \tag{3.34}$$

$$c^{ijkl} = J^{-1}F^i{}_\alpha F^j{}_\beta \mathrm{C}^{\alpha\beta\gamma\delta}F^k{}_\gamma F^l{}_\delta, \tag{3.35}$$

$$\gamma^{ij} = F^i{}_\alpha \Gamma^{\alpha\beta}F^j{}_\beta, \tag{3.36}$$

$$c_\eta = C_\eta, \tag{3.37}$$

and

$$\mathbf{b}^{ijkl} = \mathbf{c}^{ijkl} + \tfrac{1}{2}[\sigma^{ik}\delta^{jl} + \sigma^{il}\delta^{jk} + \sigma^{jk}\delta^{il} + \sigma^{jl}\delta^{ik}] - \sigma^{ij}\delta^{kl}. \tag{3.38}$$

Just as in the Lagrangian picture, these differential constitutive relations derive from an Eulerian equation of state u related to U through the identification

$$u(f^i{}_\alpha, \eta) = U(E_{\alpha\beta}, S). \tag{3.39}$$

Thus the energy is

$$\varepsilon = u(f^i{}_\alpha, \eta). \tag{3.40}$$

Simple manipulations show that

$$\sigma^{ij} = \rho \frac{\partial u}{\partial f_{i\alpha}} f^j{}_\alpha \tag{3.41}$$

and

$$\theta = \frac{\partial u}{\partial \eta}, \tag{3.42}$$

and that the coefficients $\mathbf{c}^{ijkl}$, γ^{ij}, and c_η are expressible as second derivatives of u.

With the definitions (3.40)–(3.42) and (3.29), Eqs. (3.12)–(3.14) and (3.22) comprise a complete set of conservation laws for elastic materials in the Eulerian picture.

3c. *Jump Conditions*

Suppose that a surface of discontinuity in the ambient space $\mathscr{S}$ has normal n_i and normal propagation speed s_n. Then the Rankine–Hugoniot jump conditions corresponding to Eqs. (3.8)–(3.10) and (3.21) are

$$\Delta\left[\rho\left(-s_n + v^j n_j\right)\right] = 0, \tag{3.43}$$

$$\Delta\left[\rho v^i\left(-s_n + v^j n_j\right)\right] = \Delta\sigma^{ij} n_j, \tag{3.44}$$

$$\Delta\left[\rho\left(\tfrac{1}{2}v_i v^i + \varepsilon\right)\left(-s_n + v^j n_j\right)\right] = \Delta\left(v_i \sigma^{ij}\right) n_j - \Delta q^k n_k, \tag{3.45}$$

and

$$\Delta\left[\rho f^i{}_\alpha\left(-s_n + v^j n_j\right)\right] = \Delta\left(\rho v^i f^j{}_\alpha\right) n_j. \tag{3.46}$$

To see that these Eulerian jump conditions are equivalent to the Lagrangian jump conditions, we first cast them into another form. Let $m = \overline{\rho\left(s_n - v^i n_i\right)}$. Then according to Eq. (3.43),

$$\rho_-\left(s_n - v^i_- n_i\right) = m = \rho_+\left(s_n - v^i_+ n_i\right); \tag{3.47}$$

furthermore, the jump relations (3.44)–(3.46) are equivalent to

$$-m\Delta v^i = \Delta\sigma^{ij} n_j, \tag{3.48}$$

$$-m\Delta\left(\tfrac{1}{2}v_i v^i + \varepsilon\right) = \Delta\left(v_i \sigma^{ij}\right) n_j - \Delta q^k n_k, \tag{3.49}$$

and

$$-m\Delta f^i{}_\alpha = \Delta\left(\rho v^i f^j{}_\alpha\right) n_j. \tag{3.50}$$

Assuming that the Eulerian fluid variables are related to Lagrangian variables through the definitions (3.3)–(3.7), the Lagrangian jump conditions, Eqs. (2.37)–(2.40), follow from Eqs. (3.47)–(3.50). This is because of the Piola identity $n_i\, d\mathbf{a} = JN_\alpha (F^{-1})^\alpha{}_i\, d\mathbf{A}$ and the correspondence $m\, d\mathbf{a} = M\, d\mathbf{A}$ between the mass flows across the discontinuity. To show, conversely, that the Eulerian jump conditions imply the Lagrangian conditions, we must assume that

$$\Delta\left(\rho f^i{}_\alpha\right) n_i = 0. \tag{3.51}$$

This assumption is the jump condition corresponding to Eq. (3.26), which holds if $f^i{}_\alpha$ and ρ derive from a deformation as in Theorem 1. It also follows from the Eulerian jump conditions if $s_n \neq 0$: multiplying Eq. (3.46) by n_i implies that $-s_n \Delta(\rho f^i{}_\alpha) n_i = 0$. As a consequence of assumption

(3.51), the vector N_α, as defined by

$$N_\alpha \, d\mathbf{A} = \rho n_i f^i{}_\alpha \, d\mathbf{a}, \tag{3.52}$$

suffers no jump discontinuity, in accordance with the Piola identity. With the definition $m\,d\mathbf{a} = M\,d\mathbf{A}$, Eqs. (3.47)–(3.50) now imply Eqs. (2.37)–(2.40).

Just as in gas dynamics, the jump conditions may be solved by finding a relation among the thermodynamic variables (the Hugoniot relation) together with an expression for the velocity in terms of these variables. Multiplying Eq. (3.50) by $d\mathbf{a}$, and using Eqs. (3.51)–(3.52) and $m\,d\mathbf{a} = M\,d\mathbf{A}$, yields

$$-M\Delta f^i{}_\alpha \, d\mathbf{A} = \Delta v^i N_\alpha \, d\mathbf{A}. \tag{3.53}$$

The component of this equation in the direction of N_α may be solved for the velocity jump:

$$\Delta v^i = -M\Delta f^i{}_\alpha N^\alpha / \left(N_\beta N^\beta\right). \tag{3.54}$$

If we introduce

$$\tau^i = \rho^{-1} \frac{n_k b^{ki}}{n_l b^{lm} n_m}, \tag{3.55}$$

where

$$b^{ij} = \rho_{\mathrm{ref}}^2 f^i{}_\alpha f^{j\alpha} = F^i{}_\alpha F^{j\alpha} \tag{3.56}$$

is the *Finger strain tensor*, then Eq. (3.54) may be written

$$\Delta v^i = -m\Delta\tau^i. \tag{3.57}$$

Combining this relation with Eq. (3.50) yields

$$m\Delta f^i{}_\alpha = m\Delta\tau^i \overline{\rho f^j{}_\alpha} n_j. \tag{3.58}$$

Notice that $\tau^i n_i = \rho^{-1}$, and that $\Delta f^i{}_\alpha = \Delta\rho^{-1}\overline{\rho f^i{}_\alpha} + \overline{\rho^{-1}}\Delta(\rho f^i{}_\alpha)$; thus the n_i component of Eq. (3.58) recovers assumption (3.51). Similarly, Eq. (3.57) in Eq. (3.48) results in

$$m^2\Delta\tau^i = \Delta\sigma^{ij} n_j; \tag{3.59}$$

in particular, the component of this equation in the n_i direction is

$$m^2\Delta\rho^{-1} = n_i \Delta\sigma^{ij} n_j. \tag{3.60}$$

Finally, notice that Eq. (3.48), when multiplied by $\bar{v}_i$, may be used in Eq. (3.49) to obtain

$$-m\Delta\varepsilon = \Delta v_i \overline{\sigma^{ij}} n_j - \Delta q^k n_k. \tag{3.61}$$

Therefore Eq. (3.57) shows that

$$m\left\{\Delta\varepsilon - \Delta\tau_i \overline{\sigma^{ij}} n_j\right\} = \Delta q^k n_k. \tag{3.62}$$

In the special case of gas dynamics, where $\sigma^{ij} = -P\delta^{ij}$, P being the pressure, Eq. (3.60) becomes the equation

$$m^2 \Delta\rho^{-1} = -\Delta P \tag{3.63}$$

for the Rayleigh line, while Eq. (3.62) reduces to the Hugoniot relation

$$\Delta\varepsilon + \Delta\rho^{-1}\overline{P} = 0. \tag{3.64}$$

To solve the jump conditions, notice that $\rho f^j{}_\alpha n_j$ is the same on both sides of the jump, so that Eq. (3.58) expresses all of the components of $f^i{}_\alpha$ in terms of the three variables τ^i. Therefore $f^i{}_\alpha$ may be eliminated in favor of τ^i in the expressions for ε, σ^{ij}, and θ given by the equation of state. As a result, Eqs. (3.59) and (3.62) may be solved for τ^i and η, just as the Rayleigh line equation and the Hugoniot relation may be solved for ρ^{-1} and η in gas dynamics. Thus the Eqs. (3.58), (3.59), and (3.62) comprise a complete set of jump conditions for the thermodynamic variables $f^i{}_\alpha$ and η. Once these equations have been solved, Eqs. (3.47) and (3.57) may be used to determine the shock speed and the velocities. This formulation is the starting point for an analysis of the Riemann problem for elasticity in the Eulerian picture (cf. Ref. [6]).

4. Summary

We conclude by summarizing the complete set of conservation laws governing elastic flow in the Eulerian picture;

$$\frac{\partial\rho}{\partial t} + \left(\rho v^i\right)_{;i} = 0, \tag{4.1}$$

$$\frac{\partial}{\partial t}\left(\rho v^i\right) + \left(\rho v^i v^j\right)_{;j} = \sigma^{ij}{}_{;j}, \tag{4.2}$$

$$\frac{\partial}{\partial t}\left[\rho\left(\tfrac{1}{2}v_i v^i + \varepsilon\right)\right] + \left[\rho\left(\tfrac{1}{2}v_i v^i + \varepsilon\right)v^j\right]_{;j} = \left(v_i\sigma^{ij}\right)_{;j} - q^k{}_{;k}, \tag{4.3}$$

and

$$\frac{\partial}{\partial t}\left(\rho f^i{}_\alpha\right) + \left(\rho f^i{}_\alpha v^j\right)_{;j} = \left(\rho v^i f^j{}_\alpha\right)_{;j} \tag{4.4}$$

govern the evolution of the variables $\rho = J^{-1}\rho_{\text{ref}}$, v^i, η, and $f^i{}_\alpha = \rho_{\text{ref}}^{-1} F^i{}_\alpha$. These equations are supplemented by the constitutive relations

$$\varepsilon = u\left(f^i{}_\alpha, \eta\right), \tag{4.5}$$

$$\sigma^{ij} = \rho \frac{\partial u}{\partial f_{i\alpha}} f^j{}_\alpha, \tag{4.6}$$

$$\theta = \frac{\partial u}{\partial \eta}, \tag{4.7}$$

and

$$q^i = -\kappa \theta^{;i}. \tag{4.8}$$

The corresponding jump conditions may be reduced to

$$\rho_-\left(s_n - v^i_- n_i\right) = m = \rho_+\left(s_n - v^i_+ n_i\right), \tag{4.9}$$

$$\Delta v^i = -m \Delta \tau^i, \tag{4.10}$$

$$m \Delta f^i{}_\alpha = m \Delta \tau^i \overline{\rho f^j{}_\alpha} n_j, \tag{4.11}$$

$$m^2 \Delta \tau^i = \Delta \sigma^{ij} n_j, \tag{4.12}$$

and

$$m\left\{\Delta \varepsilon - \Delta \tau_i \overline{\sigma^{ij}} n_j\right\} = \Delta q^k n_k, \tag{4.13}$$

where $\tau^i = \rho^{-1} n_k b^{ki} / (n_l b^{lm} n_m)$ and $b^{ij} = \rho_{\text{ref}}^2 f^i{}_\alpha f^{j\alpha}$.

The equations above give a complete description of elastic materials in the Eulerian picture. The incorporation of plasticity into this framework will be addressed in a follow-on paper.

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