On the Reflexivity of Point Sets

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Abstract

We introduce a new measure for planar point sets S that captures a combinatorial distance that S is from being a convex set: The *reflexivity* $\rho(S)$ of S is given by the smallest number of reflex vertices in a simple polygonalization of S. We prove combinatorial bounds on the reflexivity of point sets and study some closely related quantities, including the *convex cover* number $\kappa_c(S)$ of a planar point set, which is the smallest number of convex chains that cover S, and the *convex partition* number $\kappa_p(S)$, which is given by the smallest number of convex chains with pairwise-disjoint convex hulls that cover S.

1 Introduction

In this paper, we study a fundamental combinatorial property of a discrete set, S, of points in the plane: What is the minimum number, $\rho(S)$, of *reflex vertices* among all of the *simple polygonalizations* of S? A *polygonalization* of S is a closed tour on S whose straight-line embedding in the plane defines a connected cycle without crossings, i.e., a simple polygon. A vertex of a simple polygon is *reflex* if it has interior angle greater than π. We refer to $\rho(S)$ as the *reflexivity* of S. We let $\rho(n)$ denote the maximum possible value of $\rho(S)$ for a set S of n points.

In general, there are many different polygonalizations of a point set S. There is always at least one: simply connect the points in angular order about some point interior to the convex hull of S (e.g., the center of mass suffices). A set S has precisely one polygonalization if and only if it is in convex position; in general, though, a point set has numerous polygonalizations. Studying the set of polygonalizations (e.g., counting them, enumerating them, or generating a random element) is a challenging and active area of investigation in computational geometry [4, 5, 9, 17, 19, 34].

The reflexivity $\rho(S)$ quantifies, in a combinatorial sense, the degree to which the set of points S is in convex position. See Figure 1 for an example. We remark that there are other notions of combinatorial "distance" from convexity of a point set S, e.g., the minimum number of points to delete from

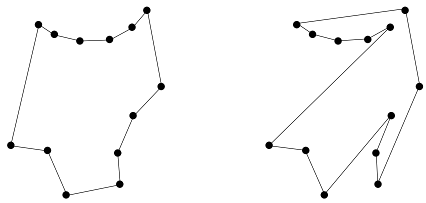

Fig. 1. Two polygonalizations of a point set, one (left) using 7 reflex vertices and one (right) using only 3 reflex vertices.

S in order that the remaining point set is in convex position, the number of convex layers, or the minimum number of changes in the orientation of triples of points of S in order to transform S into convex position.

We have conducted a formal study of reflexivity, both in terms of its combinatorial properties and in terms of an algorithmic analysis of the complexity of computing it, exactly or approximately. Some of our attention is focussed on the closely related *convex cover number* of S, which gives the minimum number of convex chains (subsets of S in convex position) that are required to cover all points of S. For this question, we distinguish between two cases: The *convex cover number*, $\kappa_c(S)$, is the smallest number of convex chains required to cover S; the *convex partition number*, $\kappa_p(S)$, is the smallest number of convex chains with pairwise-disjoint convex hulls required to cover S. Note that nested chains are feasible for a convex cover but not for a convex partition.

Motivation. In addition to the fundamental nature of the questions and problems we address, we are also motivated to study reflexivity for several other reasons:

(1) An application motivating our original investigation is that of meshes of low stabbing number and their use in performing ray shooting efficiently. If a point set S has low reflexivity or a low convex partition number, then it has a triangulation of low stabbing number, which may be much lower than the general $O(\sqrt{n})$ upper bound guaranteed to exist ([1, 20, 33]). For example, if the reflexivity is $O(1)$, then S has a triangulation with stabbing number $O(\log n)$.

(2) Classifying point sets by their reflexivity may give us some structure for dealing with the famously difficult question of counting and exploring the set of all polygonalizations of S. See [19, 34] for some references to this problem.

(3) There are several applications in computational geometry in which the number of reflex vertices of a polygon can play an important role in the complexity of algorithms. If one or more polygons are *given* to us, there are many problems for which more efficient algorithms can be written with complexity in terms of "r" (the number of reflex vertices), instead of "n" (the total number of vertices), taking advantage of the possibility that we may have $r \ll n$

for some practical instances (see, e.g., [21, 25]). The number of reflex vertices also plays an important role in convex decomposition problems for polygons; see Keil [26] for a recent survey, and see Agarwal, Flato, and Halperin [2] for applications of convex decompositions to computing Minkowski sums of polygons.

(4) Reflexivity is intimately related to the issue of convex cover numbers, which has roots in the classical work of Erdős and Szekeres [15, 16], and has been studied more recently by Urabe et al. [23, 24, 30, 31].

(5) Our problems are related to some problems in curve (surface) reconstruction, where the goal is to obtain a "good" polygonalization of a set of sample points (see, e.g., [7, 12, 13]).

Related Work. The study of convex chains in finite planar point sets is the topic of classical papers by Erdős and Szekeres [15, 16], who showed that any point set of size n has a convex subset of size $t = \Omega(\log n)$. This is closely related to the convex cover number κ_c, since it implies an asymptotically tight bound on $\kappa_c(n)$, the worst-case value for sets of size n. There are still a number of open problems related to the exact relationship between t and n; see, for example, [28] for recent developments.

Other issues have been considered, such as the existence and computation ([14]) of large "empty" convex subsets (i.e., with no points of S interior to their hull); this is related to the convex partition number, $\kappa_p(S)$. It was shown by Horton [22] that there are sets with no empty convex chain larger than 6; this implies that $\kappa_p(n) \geq n/6$.

Tighter worst-case bounds on $\kappa_p(n)$ were given by Urabe [30, 31], who shows that $\lceil (n-1)/4 \rceil \leq \kappa_p(n) \leq \lceil 2n/7 \rceil$ and that $\kappa_c(n) = \Theta(n/\log n)$ (with the upper and lower bounds having a gap of roughly a factor of 2). (Urabe [32] also studies the convex partitioning problem in $\Re^3$, where, in particular, the upper bound on $\kappa_p(n)$ is shown to be $\lceil 2n/9 \rceil$.) Most recently, Hosono and Urabe [24] have obtained improved bounds on the size of a partition of a set of points into disjoint convex quadrilaterals, which has the consequence of improving the upper bound on $\kappa_p(n)$: $\kappa_p(n) \leq \lceil 5n/18 \rceil$ and $\kappa_p(n) \leq (3n+1)/11$ for $n = 11 \cdot 2^{k-1} - 4$ $(k \geq 1)$. The remaining gaps in the constants between upper and lower bounds for $\kappa_c(n)$ and $\kappa_p(n)$ (as well as the gap that our bounds exhibit for reflexivity in terms of n) all point to the apparently common difficulty of these combinatorial problems on convexity.

For a given set of points, we are interested in polygonalizations of the points that are "as convex as possible". This has been studied in the context of TSP (traveling salesperson problem) tours of a point set S, where convexity of S implies (trivially) the optimality of a convex tour. Convexity of a tour can be characterized by two conditions. If we drop the global condition (i.e., no crossing edges), but keep the local condition (i.e., no reflex vertices), we get "pseudo-convex" tours. In [18] it was shown that any set with $|S| \geq 5$ has such a pseudo-convex tour. It is natural to require the global condition of simplicity instead, and minimize the number of local violations – i.e., the number of reflex vertices. This kind of problem is similar to that of minimizing the total amount of turning in a tour, as studied by Aggarwal et al. [3].

The number of polygonalizations on n points is, in general, exponential in n; García et al. [19] prove a lower bound of $\Omega(4.64^n)$.

Another related problem is studied by Hosono et al. [23]: Compute a polygonalization P of a point set S such that the interior of P can be decomposed into a minimum number ($f(S)$) of empty convex polygons. They prove that $\lfloor (n-1)/4 \rfloor \leq f(n) \leq \lfloor (3n-2)/5 \rfloor$, where $f(n)$ is the maximum possible value of $f(S)$ for sets S of n points. The authors conjecture that $f(n)$ grows like $n/2$. For reflexivity $\rho(n)$, we show that $\lfloor n/4 \rfloor \leq \rho(n) \leq \lceil n/2 \rceil$ and conjecture that $\rho(n)$ grows like $n/4$, which, if true, would imply that $f(n)$ grows like $n/2$.

We mention one final related problem. A *convex decomposition* of a point set S is a convex planar polygonal subdivision of the convex hull of S whose vertices are S. Let $g(S)$ denote the minimum number of faces in a convex decomposition of S, and let $g(n)$ denote the maximum value of $g(S)$ over all n-point sets S. It has been conjectured ([29]) that $g(n) = n + c$ for some constant c, and it is known that $g(n) \leq 3n/2$ ([29]) and that $n + 2 \leq g(n)$ ([6]).

Summary of Main Results. In this paper, we prove the following results on reflexivity:

- Tight bounds on the worst-case value of $\rho(S)$ in terms of n_I, the number of points of S interior to the convex hull of S; in particular, we show that $\rho(S) \leq \lceil n_I/2 \rceil$ and that this upper bound can be achieved by a class of examples.
- Upper and lower bounds on $\rho(S)$ in terms of $n = |S|$; in particular, we show that $\lfloor n/4 \rfloor \leq \rho(n) \leq \lceil n/2 \rceil$.
- Upper and lower bounds on "Steiner reflexivity", which is defined with respect to the class of polygonalizations that allow Steiner vertices (not from the input set S).

In Section 6 we study a closely related problem – that of determining the "inflectionality" of S, defined to be the minimum number of inflection edges (joining a convex to a reflex vertex) in any polygonalization of S. We give an $O(n \log n)$ time algorithm to determine an inflectionality-minimizing polygonalization, which we show will never need more than 2 inflection edges.

Additional Results. We also summarize additional results we have obtained in this line of research; in the interest of preserving space here, the proofs of the following results appear in the extended paper [8]:

- In the case in which S has two layers, we show that $\rho(S) \leq \lceil n/4 \rceil$, and this bound is tight.

- We prove that it is NP-complete to compute the convex cover number ($\kappa_c(S)$) or the convex partition number ($\kappa_p(S)$), for a given point set S.
- We give polynomial-time approximation algorithms, having approximation factor $O(\log n)$, for the problems of computing convex cover number, convex partition number, or Steiner reflexivity of S.
- We give efficient exact algorithms to test if $\rho(S) = 1$ or $\rho(S) = 2$.

2 Preliminaries

Throughout this paper, S will be a set of n points in the plane $\Re^2$. A polygonalization, P, of S is a simple polygon whose vertex set is S. Let $\mathcal{P}$ be the set of all polygonalizations of S. Note that $\mathcal{P}$ is not empty, since any point set S having $n \geq 3$ points has at least one polygonalization (e.g., the star-shaped polygonalization obtained by sorting points of S angularly about a point interior to the convex hull of S).

Each vertex of a simple polygon P is either *reflex* or *convex*, according to whether the interior angle at the vertex is greater than π or less than or equal to π, respectively. We let $r(P)$ (resp., $c(P)$) denote the number of reflex (resp., convex) vertices of P. We define the *reflexivity* of a planar point set S to be $\rho(S) = \min_{P \in \mathcal{P}} r(P)$. Similarly, the *convexivity* of a planar point set S is defined to be $\chi(S) = \max_{P \in \mathcal{P}} c(P)$. Note that $\chi(S) = n - \rho(S)$. We let $\rho(n) = \max_{|S|=n} \rho(S)$.

We let $\mathrm{CH}(S)$ denote the *convex hull* of S. The point set S is partitioned into (convex) *layers*, $S_1, S_2, \ldots$, where the first layer is given by the set S_1 of points of S on the boundary of $\mathrm{CH}(S)$, and the ith layer, S_i ($i \geq 2$) is given by the set of points of S on the boundary of $\mathrm{CH}(S \setminus (S_1 \cup \cdots \cup S_{i-1}))$. We say that S has k *layers* or *onion depth* k if $S_k \neq \emptyset$, while $S_{k+1} = \emptyset$. We say that S is in *convex position* (or forms a *convex chain*) if it has one layer (i.e., $S = S_1$).

A *Steiner point* is a point not in the set S that may be added to S in order to improve some structure of S. We define the *Steiner reflexivity* $\rho'(S)$ to be the minimum number of reflex vertices of any simple polygon with vertex set $V \supset S$. We let $\rho'(n) = \max_{|S|=n} \rho'(S)$. A *convex cover* of S is a set of subsets of S whose union covers S, such that each subset is a convex chain (a set in convex position). A *convex partition* of S is a partition of S into subsets each of which is in convex position, such that the convex hulls of the subsets are pairwise disjoint. We define the *convex cover number*, $\kappa_c(S)$, to be the minimum number of subsets in a convex cover of S. We similarly define the *convex partition number*, $\kappa_p(S)$. We denote by $\kappa_c(n)$ and $\kappa_p(n)$ the worst-case values for sets of size n.

Finally, we state a basic property of polygonalizations of point sets.

Lemma 2.1. *In any polygonalization of S, the points of S that are vertices of the convex hull of S are convex vertices of the polygonalization, and they*

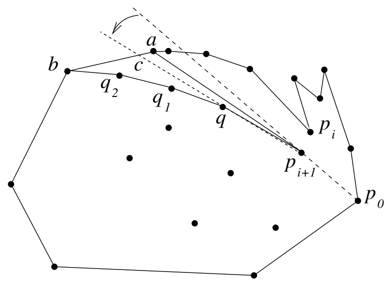

Fig. 2. Computing a polygonalization with at most $\lceil n_I/2 \rceil$ reflex vertices.

occur in the polygonalization in the same order in which they occur along the convex hull.

Proof. Any polygonalization P of S must lie within the convex hull of S, since edges of the polygonalization are convex combinations of points of S. Thus, if $p \in S$ is a vertex of CH(S), then the local neighborhood of P at p lies within a convex cone, so p must be a convex vertex of P.

Consider a clockwise traversal of P and let p and q be two vertices of CH(S) occurring consecutively along P. Then p and q must also appear consecutively along a clockwise traversal of the boundary of CH(S), since the subchain of P linking p to q partitions CH(S) into a region to its left (which is outside the polygon P) and a region to its right (which must contain all points of S not in the subchain). □

3 Combinatorial Bounds

3.1 Reflexivity

One of our main combinatorial results establishes an upper bound on the reflexivity of S that is worst-case tight in terms of the number n_I of points *interior* to the convex hull, CH(S), of S. Since, by Lemma 2.1, the points of S that are vertices of CH(S) are required to be convex vertices in any (non-Steiner) polygonalization of S, the bound in terms of n_I seems to be quite natural.

Theorem 3.1. *Let S be a set of n points in the plane, n_I of which are interior to the convex hull CH(S). Then $\rho(S) \leq \lceil n_I/2 \rceil$.*

Proof. We describe a polygonalization in which at most half of the interior points are reflex. We begin with the polygonalization of the convex hull vertices that is given by the convex polygon bounding the hull. We then iteratively incorporate interior points of S into the polygonalization. Fix a point p_0 that lies on the convex hull of S. At a generic step of the algorithm, the following invariants hold: (1) our polygonalization consists of a simple

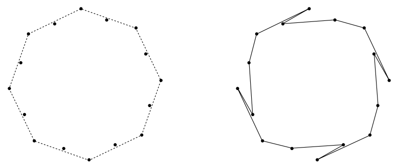

Fig. 3. Left: The configuration of points, $S_0(n)$, which has reflexivity $\rho(S_0(n)) \geq \lceil n_I/2 \rceil$. Right: A polygonalization having $\lceil n_I/2 \rceil$ reflex vertices.

polygon, P, whose vertices form a subset of S; and (2) all points $S' \subset S$ that are not vertices of P lie interior to P; in fact, the points S' all lie within the subpolygon, Q, to the left of the diagonal p_0p_i, where p_i is a vertex of P such that the subchain of ∂P from p_i to p_0 (counter-clockwise) together with the diagonal p_0p_i forms a convex polygon (Q). If S' is empty, then P is a polygonalization of S and we are done; thus, assume that $S' \neq \emptyset$. Define p_{i+1} to be the first point of S' that is encountered when sweeping the ray $\overrightarrow{p_0p_i}$ counter-clockwise about its endpoint p_0. Then we sweep the subray with endpoint p_{i+1} further counter-clockwise, about p_{i+1}, until we encounter another point, q, of S'. (If $|S'| = 1$, we can readily incorporate p_{i+1} into the polygonalization, increasing the number of reflex vertices by one.) Now the ray $\overrightarrow{p_{i+1}q}$ intersects the boundary of P at some point $c \in ab$ on the boundary of Q.

As a next step, we modify P to include interior points p_{i+1} and q (and possibly others as well) by replacing the edge ab with the chain $(a, p_{i+1}, q, q_1, \ldots, q_k, b)$, where the points q_i are interior points that occur along the chain we obtain by "pulling taut" the chain (q, c, b). In this "gift wrapping" fashion, we continue to rotate rays counter-clockwise about each interior point q_i that is hit until we encounter b. This results in incorporating at least two new interior points (of S') into the polygonalization P, while creating only one new reflex vertex (at p_{i+1}). It is easy to check that the invariants (1) and (2) hold after this step. □

In fact, the upper bound of Theorem 3.1, $\rho(S) \leq \lceil n_I/2 \rceil$, is *tight* in the worst case, as we now argue based on the special configuration of points, $S = S_0(n)$, in Figure 3. The set $S_0(n)$ is defined for any integer $n \geq 6$, as follows: $\lceil n/2 \rceil$ points are placed in convex position (e.g., forming a regular $\lceil n/2 \rceil$-gon), forming the convex hull CH(S), and the remaining $n_I = \lfloor n/2 \rfloor$ interior points are also placed in convex position, each one placed "just inside" CH(S), near the midpoint of an edge of CH(S). The resulting configuration $S_0(n)$ has two layers in its convex hull.

Lemma 3.2. *For any* $n \geq 6$, $\rho(S_0(n)) \geq \lceil n_I/2 \rceil \geq \lfloor n/4 \rfloor$.

Proof. Let $(x_1, x_2, \ldots, x_{\lceil n/2 \rceil})$ denote the points of $S_0(n)$ on the convex hull, in clockwise order, and let $(v_1, v_2, \ldots, v_{\lfloor n/2 \rfloor})$ denote the remaining points of $S_0(n)$, with v_i just inside the convex hull edge (x_i, x_{i+1}). We define $x_{\lceil n/2 \rceil + 1} = x_1$.

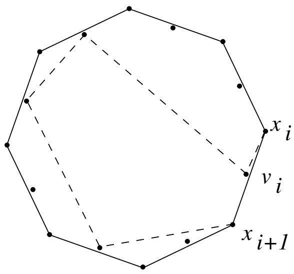

Fig. 4. Proof of the lower bound: $\rho(S_0(n)) \geq \lceil n_I/2 \rceil \geq \lfloor n/4 \rfloor$.

Consider any polygonalization, P, of $S_0(n)$. From Lemma 2.1 we know that the points x_i are convex vertices of P, occurring in the order $x_1, x_2, \ldots, x_{\lceil n/2 \rceil}$ around the boundary of P. Consider the subchain, γ_i, of ∂P that goes from x_i to x_{i+1}, clockwise around ∂P. Let m_i denote the number of points v_j, interior to the convex hull of $S_0(n)$, that appear along γ_i.

If $m_i = 0$, $\gamma_i = x_i x_{i+1}$. If $m_i = 1$, then $\gamma_i = x_i v_i x_{i+1}$ and v_i is a reflex vertex of P; to see this, note that v_i lies interior to the triangle determined by x_i, x_{i+1}, and any v_j with $j \neq i$. If $m_i > 1$, then we claim that (a) v_i must be a vertex of the chain γ_i, (b) v_i is a convex vertex of P, and (c) any other point v_j, $j \neq i$, that is a vertex of γ_i must be a reflex vertex of P. This claim follows from the fact that the points x_i, x_{i+1}, and any nonempty subset of $\{v_j : j \neq i\}$ are in convex position, with the point v_i interior to the convex hull. Refer to Figure 4, where the subchain γ_i is shown dashed.

Thus, the number of reflex vertices of P occurring along γ_i is in any case at least $\lceil m_i/2 \rceil$, and we have

$$\begin{aligned} \rho(S_0(n)) &\geq \sum \lceil m_i/2 \rceil \\ &\geq \left\lceil \sum (m_i/2) \right\rceil = \lceil n_I/2 \rceil \geq \lfloor n/4 \rfloor. \end{aligned}$$

□

Since $n_I \leq n$, the corollary below is immediate from Theorem 3.1 and Lemma 3.2. The gap in the bounds for $\rho(n)$, between $\lfloor n/4 \rfloor$ and $\lceil n/2 \rceil$, remains an intriguing open problem. While our combinatorial bounds are worst-case tight in terms of n_I (the number of points of S whose convexity/reflexivity is not forced by the convex hull of S), they are not worst-case tight in terms of n.

Corollary 3.3. $\lfloor n/4 \rfloor \leq \rho(n) \leq \lceil n/2 \rceil$.

Based on experience with a software tool developed by A. Dumitrescu that computes, in exponential time, the reflexivity of user-specified or randomly generated point sets, as well as the proven behavior of $\rho(n)$ for small values of n (see Section 3.5), we make the following conjecture:

Conjecture 3.4. $\rho(n) = \lfloor n/4 \rfloor$.

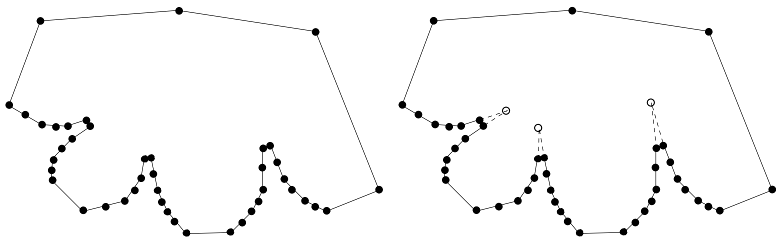

Fig. 5. Left: A point set S having reflexivity $\rho(S) = r$. Right: The reflexivity of S when Steiner points *are* permitted is substantially reduced from the no-Steiner case: $\rho'(S) = r/2$.

3.2 Steiner Points

If we allow Steiner points in the polygonalizations of S, the reflexivity of S may go down substantially, as the example in Figure 5 shows. In fact, the illustrated class of examples shows that the use of Steiner points may allow the reflexivity to go down by a factor of two. The Steiner reflexivity, $\rho'(S)$, of S is the minimum number of reflex vertices of any simple polygon with vertex set $V \supset S$. We conjecture that $\rho'(S) \geq \rho(S)/2$ for any set S, which would imply that this class of examples (essentially) maximizes the ratio $\rho(S)/\rho'(S)$.

Conjecture 3.5. *For any set S of points in the plane, $\rho'(S) \geq \rho(S)/2$.*

We have seen (Corollary 3.3) that $\lfloor n/4 \rfloor \leq \rho(n) \leq \lceil n/2 \rceil$. We now show that allowing Steiner points in the polygonalization allows us to prove a smaller upper bound, while still being able to prove roughly the same lower bound:

Theorem 3.6.

$$\left\lceil \frac{n-1}{4} \right\rceil - 1 \leq \rho'(n) \leq \left\lceil \frac{n}{3} \right\rceil .$$

Proof. For the upper bound, we give a specific method of constructing a polygonalization (with Steiner points) of a set S of n points. Sort the points S by their x-coordinates and group them into consecutive triples. Let p_{n+1} denote a (Steiner) point with a very large positive y-coordinate and let p_0 denote a (Steiner) point with a very negative y-coordinate. Each triple, together with either point p_{n+1} or point p_0, forms a convex quadrilateral. Then, we can polygonalize S using one reflex (Steiner) point per triple, as shown in Figure 6, placed very close to p_{n+1} or p_0 accordingly. This polygonalization has at most $\lceil n/3 \rceil$ reflex points.

For the lower bound, we consider the configuration of n points, S, used in Urabe [30] to prove that $\kappa_p(n) \geq \lceil (n-1)/4 \rceil$. For this set S of n points, let P be a Steiner polygonalization having r reflex vertices. Then the simple polygon P can be partitioned into $r+1$ (pairwise-disjoint) convex pieces;

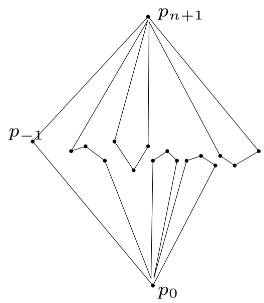

Fig. 6. Polygonalization of n points using only $\lceil n/3 \rceil$ reflex (Steiner) points.

this is a simple observation of Chazelle [10] (see Theorem 2.5.1 of [27]). The points S occur as a subset of the vertices of these pieces; thus, the partitioning also decomposes S into at most $r+1$ subsets, each in convex position. Since $\kappa_p(n) \geq \lceil (n-1)/4 \rceil$, we get that $r \geq \lceil (n-1)/4 \rceil - 1$. □

3.3 Two-Layer Point Sets

Let S be a point set that has two (convex) layers. It is clear from our repeated use of the example in Figure 3 that this is a natural case that is a likely candidate for worst-case behavior. With a very careful analysis of this case, we are able to obtain tight combinatorial bounds on the worst-case reflexivity in terms of n. The proof of the following theorem is quite technical and long, so it is deferred to the extended paper [8].

Theorem 3.7. *Let S be a set of n points having two layers. Then $\rho(S) \leq \lceil n/4 \rceil$, and this bound is tight in the worst case.*

Remark. Using a variant of the polygonalization given in the proof of Theorem 3.7, it is possible to show that a two-layer point set S in fact has a polygonalization with at most $\lceil n/3 \rceil$ reflex vertices such that none of the edges in the polygonalization pass through the interior of the convex hull of the second layer. (The polygonalization giving upper bound of $\lceil n/4 \rceil$ requires edges that pass through the interior of the convex hull of the second layer.) This observation may be useful in attempts to reduce the worst-case upper bound ($\rho(n) \leq \lceil n/2 \rceil$) for more general point sets S.

3.4 Convex Cover/Partition Numbers

As a consequence of the Erdős-Szekeres theorem [15, 16], Urabe has given bounds on the convex cover number of a set of n points:

$$\frac{n}{\log_2 n + 2} < \kappa_c(n) < \frac{2n}{\log_2 n - \log_2 e} .$$

Urabe [30] and Hosono and Urabe [24] have obtained bounds as well on the convex partition number of an n-point set:

$$\left\lceil \frac{n-1}{4} \right\rceil \le \kappa_p(n) \le \left\lceil \frac{5n}{18} \right\rceil .$$

While it is trivially true that $\kappa_c(S) \le \kappa_p(S)$ the ratio $\kappa_p(S)/\kappa_c(S)$ for a set S may be as large as $\Theta(n)$; the set $S = S_0(n)$ (Figure 3) has $\kappa_c(S) = 2$, but $\kappa_p(S) \ge n/4$.

The fact that $\kappa_p(S) \le \rho(S) + 1$ follows easily by iteratively adding $\rho(S)$ segments to an optimal polygonalization P, bisecting each reflex angle. The result is a partitioning of P into $\rho(S)+1$ convex pieces. Thus, we can obtain a convex partitioning of S by associating a subset of S with each convex piece of P, assigning each point of S to the subset associated with any one of the convex pieces that has the point on its boundary.

We believe that the relationship between reflexivity ($\rho(S)$) and convex partition number ($\kappa_p(S)$) goes the other way as well: A small convex partition number should imply a small reflexivity. In particular, we have invested considerable effort in trying to prove the following conjecture:

Conjecture 3.8. $\rho(S) = O(\kappa_p(S))$.

The reflexivity can be as large as *twice* the convex cover number ($\rho(S) = 2\kappa_p(S)$), as illustrated in the example of Figure 7; however, this is the worst class of examples we have found so far.

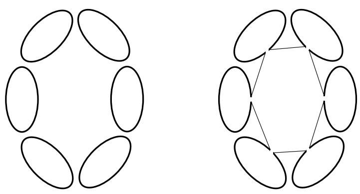

Fig. 7. An example with $\rho(S) = 2\kappa_p(S)$. Each thick oval shape represents a numerous subset of points of S in convex position.

Turning briefly to Steiner reflexivity, it is not hard to see that $\rho'(S) = O(\kappa_p(S))$ (see [8]). Thus, a proof of Conjecture 3.8 would follow from the validity of Conjecture 3.5.

3.5 Small Point Sets

It is natural to consider the exact values of $\rho(n)$, $\kappa_c(n)$, and $\kappa_p(n)$ for small values of n. Table 1 below shows some of these values, which we obtained through (sometimes tedious) case analysis. Aichholzer and Krasser [5] have recently applied their software that enumerates point sets of size n of all distinct order types to verify our results computationally; in addition, they

Table 1. Worst-case values of ρ, κ_c, κ_p for small values of n.

n	$\rho(n)$	$\kappa_c(n)$	$\kappa_p(n)$
≤ 3	0	1	1
4	1	2	2
5	1	2	2
6	2	2	2
7	2	2	2
8	2	2	3
9	3	3	3
10	3	–	–

have obtained the result that $\rho(10) = 3$. (Experiments are currently under way for $n = 11$; values of $n \geq 12$ seem to be intractable for enumeration.)

4 Complexity

In the extended paper [8], we prove lower bounds on the complexity of computing the convex cover number, $\kappa_c(S)$, and the convex partition number, $\kappa_p(S)$. The proof for the convex cover number uses a reduction of the problem 1-in-3 SAT and is inspired by the hardness proof for the ANGULAR METRIC TSP given in [3]. The proof for the convex partition number uses a reduction from PLANAR 3 SAT.

Theorem 4.1. *It is NP-complete to decide whether for a planar point set S the convex cover number $\kappa_c(S)$ or the convex partition number $\kappa_p(S)$ is below some threshold k.*

So far, the complexity status of determining the reflexivity of a point set remains open. However, the apparently close relationship between convex cover/partition numbers and reflexivity leads us to believe the following:

Conjecture 4.2. *It is NP-complete to determine the reflexivity $\rho(S)$ of a point set.*

5 Algorithms

We have obtained a number of algorithmic results on computing, exactly or approximately, reflexivity and convex cover/partition numbers. We summarize the results here, but we defer the proofs of the theorems to the extended paper [8].

Theorem 5.1. *Given a set S of n points in the plane, in $O(n \log n)$ time one can compute a polygonalization of S having at least $\chi(S)/2$ convex vertices, where $\chi(S) = n - \rho(S)$ is the convexivity of S.*

Theorem 5.2. *Given a set S of n points in the plane, the convex cover number $\kappa_c(S)$, the convex partition number $\kappa_p(S)$, and the Steiner reflexivity $\rho'(S)$ can each be computed approximately, within a factor $O(\log n)$, in polynomial time.*

For small values of r, we have devised particularly efficient algorithms that check if $\rho(S) \leq r$ and, if so, produce a witness polygonalization having at most r vertices. Of course, the case $r = 0$ is trivial, since that is equivalent to testing if S lies in convex position (which is readily done in $O(n \log n)$ time, which is worst-case optimal). One can obtain an $n^{O(r)}$ algorithm by enumerating over all combinatorially distinct (with respect to S) convex subdivisions of CH(S) into $O(r)$ convex faces, testing that the subsets of S within each face are in convex position, and then checking all possible ways to order these $O(r)$ convex chains to form a circuit that may form a simple polygon. With a careful analysis of the cases $r = 1, 2$, we show that testing if $\rho(S) = 1$ can be done in time $O(n \log n)$, which we prove is worst-case optimal, and that testing if $\rho(S) = 2$ can be done in $O(n^3 \log n)$ time. See [8] for details.

6 Inflectionality of Point Sets

Consider a clockwise traversal of a polygonalization, P, of S. Then, convex (resp., reflex) vertices of P correspond to *right* (resp., *left*) turns. In computing the reflexivity of S we desire a polygonalization that minimizes the number of left turns. In this section we consider the related problem in which we want to minimize the number of *changes* between left-turning and right-turning during a traversal that starts (and ends) at a point interior to an edge of P. We define the minimum number of such transitions between left and right turns to be the *inflectionality*, $\phi(S)$, of S, where the minimum is taken over all polygonalizations of S. Clearly, $\phi(S)$ must be an even integer; it is zero if and only if S is in convex position. Somewhat surprisingly, it turns out that $\phi(S)$ can only take on the values 0 or 2:

Theorem 6.1. *For any finite set S of n points in the plane, $\phi(S) \in \{0, 2\}$, with $\phi(S) = 0$ precisely when S is in convex position. In $O(n \log n)$ time, one can determine $\phi(S)$ as well as a polygonalization that achieves inflectionality $\phi(S)$.*

Proof. If S is in convex position, then trivially $\phi(S) = 0$. Thus, assume that S is not in convex position. Then $\phi(S) \neq 0$, so $\phi(S) \geq 2$. We claim that $\phi(S) = 2$. For simplicity, we assume that S is in general position.

Consider the ℓ nested convex polygons, $C_1, C_2, \ldots, C_\ell$, whose boundaries constitute the ℓ layers (the "onion") of the set S; these can be computed in time $O(n \log n)$ [11].

We construct a "spiral" polygonalization of S based on taking one edge, ab, of C_1, and replacing it with a pair of right-turning chains from a to $p \in S \cap C_\ell$ and from b to p. The two chains exactly cover the points of S on layers $C_2, \ldots, C_\ell$. A constructive proof that such a polygonalization exists is based on the following claim:

Claim 6.2. *For any* $1 \leq m < \ell$ *and any pair,* $a, b \in S$*, of vertices of* C_m*, there exist two purely right-turning chains,* $\gamma_a = (a, u_1, u_2, \ldots, u_i, p)$ *and* $\gamma_b = (b, v_1, v_2, \ldots, v_j, p)$*, such that the points of* S *interior to* C_m *are precisely the set* $\{u_1, u_2, \ldots, u_i, p, v_i, v_2, \ldots, v_j\}$*.*

Proof of Claim. We prove the claim by induction on m. If $m = \ell - 1$, the claim follows easily, by a case analysis as illustrated in Figure 8.

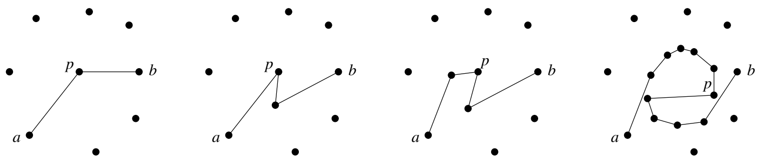

Fig. 8. Simple case in the inductive proof: $m = \ell - 1$. There are four subcases, left to right: (i) C_ℓ is a single point; (ii) C_ℓ is a line segment determined by two points of S; (iii) C_ℓ is a triangle determined by three points of S; or (iv) C_ℓ is a convex polygon whose boundary contains four or more points of S.

Assume that the claim holds for $m \geq k+1$ and consider the case $m = k$. If C_{k+1} is either a single vertex or a line segment (which can only happen if $k + 1 = \ell$), the claim trivially follows; thus, we assume that C_{k+1} has at least three vertices. We let u_1 be the vertex of C_{k+1} that is a *left tangent* vertex with respect to a (meaning that C_{k+1} lies in the closed halfplane to the right of the oriented line au_1); we let v be the left tangent vertex of C_{k+1} with respect to b. Refer to Figure 9. If $v = u_1$, we define v_1 to be the vertex of C_{k+1} that is the counter-clockwise neighbor of u_1; otherwise, we let $v_1 = v$. Let a' be the counter-clockwise neighbor of v_1. Let b' be the counter-clockwise neighbor of u_1. (Thus, b' may be the same point as v_1.) By the induction hypothesis, we know that there exist right-turning chains, $\gamma_{a'}$ and $\gamma_{b'}$, starting from the points a' and b', spiraling inwards to a point p interior to C_{k+1}. Then we construct γ_a to be the chain from a to u_1, around the boundary of C_{k+1} clockwise to a', and then along the chain $\gamma_{a'}$. Similarly, we construct γ_b to be the chain from b to v_1, around the boundary of C_{k+1} clockwise to b', and then along the chain $\gamma_{b'}$. □

The proof of the above claim is constructive; the required chains are readily obtained in $O(n \log n)$ time, given the convex layers. This concludes the proof of the theorem. □

7 Open Problems

There are a number of interesting open problems that our work suggests. First, there are the four specific conjectures mentioned throughout the paper;

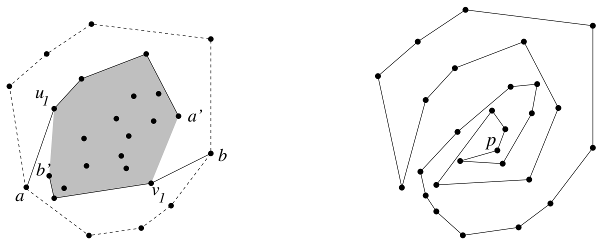

Fig. 9. Left: Constructing the spiraling chains γ_a and γ_b. Right: An example of the resulting spiral polygonalization.

these represent to us the most outstanding open questions raised by our work. In addition, we mention two other areas of future study:

1. Instead of minimizing the number of reflex vertices, can we compute a polygonalization of S that minimizes the sum of the *turn angles* at reflex vertices? (The turn angle at a reflex vertex having interior angle $\theta > \pi$ is defined to be $\theta - \pi$.) This question was posed to us by Ulrik Brandes. It may capture a notion of goodness of a polygonalization that is useful for curve reconstruction. The problem differs from the angular metric TSP ([3]) in that the only turn angles contributing to the objective function are those of reflex vertices.

2. What can be said about the generalization of the reflexivity problem to polyhedral surfaces in three dimensions? This may be of particular interest in the context of surface reconstruction.

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