

AMS 569 - Homework #5

Section 6.7

6.4 (extra credit). This is a somewhat odd problem. Let ϕ be the characteristic function of both X and Y . We have: $2X = (X + Y) + (X - Y)$. By hypothesis $X + Y$ and $X - Y$ are independent, so: $\phi(2t) = \phi_{2X}(t) = \phi_{X+Y}(t)\phi_{X-Y}(t) = \phi_X(t)\phi_Y(t)\phi_X(t)\phi_Y(-t)$ (since also X and Y are independent). We get then the following functional equation for ϕ :

$$\phi(2t) = \phi(t)^3\phi(\bar{t})$$

Since ϕ is continuous near $t = 0$, and $\phi(0) = 1$, we get that ϕ is nonzero in a neighborhood $[-\epsilon, +\epsilon]$. Near zero, we can write $\phi(t)$ as:

$$\phi(t) = e^{a(t)+ib(t)}$$

with $a(0) = 0, b(0) = 0$. Plugging this in the functional equation, we get:

$$\begin{aligned} a(2t) &= 4a(t) \\ b(2t) &= 2b(t) \end{aligned}$$

We also know that X has a second moment, this means that ϕ is twice differentiable, and therefore both $a(t)$ and $b(t)$ are twice differentiable.

By the way: what do we want to prove? $a(t) = -\frac{t^2}{2}; b(t) = 0!!!!$

This is what we get if we take second derivatives:

$$\begin{aligned} a''(2t) &= a''(t) \\ 2b''(2t) &= b''(t) \end{aligned}$$

We claim that $b''(t)$ has to be equal to zero: if it was n't there would be a point t where $b''(t)$ is not zero, but then at $t/2$ is would two times as large, and at $\frac{t}{2^n}$ it would be 2^n times as large. Let $n \rightarrow \infty$, then b'' would be unbounded at 0, which contradicts continuity of b'' at 0. We claim that a'' has to be constant, because if it were not constant over some interval, it would be equally non constant over the corresponding interval half as close to zero, iterating this thought, we get that $\limsup_{t \rightarrow 0} a''(t) \neq \liminf_{s \rightarrow 0} a''(t)$ contradicting continuity of a'' at zero.

Therefore $a(t) = \alpha t + \beta t^2$. The requirement that X have mean zero means that $\alpha = 0$. The requirement that X have variance one means that $\beta = -\frac{1}{2}$.

6.7.

- a) Let X_i be iid and let N have the $P(\lambda)$ distribution, and independent of the X_i let $Z = \sum_{i=1}^N X_i$. Prove that $\phi_Z = e^{\lambda(\phi_X - 1)}$

Proof:

$$\begin{aligned} \phi_Z(t) &= E(e^{itZ}) = \sum_{n=0}^{\infty} E(e^{itZ|N=n})P(N=n) \\ &= \sum_{n=0}^{\infty} \phi_X(t)^n e^{-\lambda} \frac{\lambda^n}{n!} \\ &= e^{-\lambda} e^{\lambda\phi_X(t)} \end{aligned}$$

- b) We need to compute $\frac{d}{dt}\phi_Z(t) = e^{-\lambda} e^{\lambda(\phi_X(t))} \lambda \frac{d}{dt}\phi_X(t)$ and $\frac{d^2}{dt^2}\phi_Z(t) = e^{-\lambda} e^{\lambda(\phi_X(t))} (\lambda \frac{d}{dt}\phi_X(t))^2 + e^{-\lambda} e^{\lambda(\phi_X(t))} \lambda \frac{d^2}{dt^2}\phi_X(t)$ Evaluation at $t = 0$ yields $E(Z) = \lambda E(X)$, and $E(Z^2) = \lambda^2 E(X)^2 + \lambda E(X^2)$, and therefore $Var(Z) = E(Z^2) - E(Z)^2 = \lambda E(X^2)$.

6.8. If $Y = Y_\lambda$ has the $P(\lambda)$ distribution, then $\phi_Y(t) = e^{\lambda(e^{it}-1)}$. Therefore $Z = \frac{Y-\lambda}{\sqrt{\lambda}}$ has as characteristic function $\phi_Z(t) = E(e^{it\frac{Y-\lambda}{\sqrt{\lambda}}}) = e^{-it\sqrt{\lambda}}\phi_Y(t\sqrt{\lambda})$. As $\lambda \rightarrow \infty$, for all t : $t/\sqrt{\lambda} \rightarrow 0$. We get $\phi_Z(t) = e^{-it\sqrt{\lambda}}e^{\lambda(it/\sqrt{\lambda} - (t/\sqrt{\lambda})^2) + o(\lambda^{-1/2})}$, so as $\lambda \rightarrow \infty$ $\phi_Z(t) \rightarrow e^{-t^2/2}$. By theorem 6.15, Z_λ then tends to standard normal as $\lambda \rightarrow \infty$.

6.15. Let U_i be iid with distribution $E(\lambda)$.

- a) $\phi_U(t) = \int_0^\infty e^{itx} \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda - it}$. Its derivative at $t = 0$ equals it/λ , so $E(U) = \frac{t}{\lambda}$.
- b) Let $T_k = \sum_{i=1}^k U_i$. Then $\phi_{T_k} = (\frac{\lambda}{\lambda - it})^k$, because the U_i are independent. From the table on page 164, we conclude that T_k has the Gamma distribution with $a = k$, λ , i.e. its pdf is $\frac{1}{(k-1)!} \lambda^k e^{-\lambda t} t^{k-1}$, ($t > 0$).
- c) Note we have a typo here! The event $N_t = k$ means that by time t , we have exactly k events. Therefore if $k \geq 1$: $P(N_t = k) = F_{T_k}(t) - F_{T_{k+1}}(t)$, while $P(N_t = 0) = e^{-\lambda t}$
- d) Fix t , the characteristic function for N_t , $\phi_{N_t}(u)$ equals

$$\begin{aligned} e^{-\lambda t} + \sum_{k=1}^{\infty} e^{iuk} (F_{T_k}(t) - F_{T_{k+1}}(t)) &= \\ e^{-\lambda t} + \sum_{k=1}^{\infty} e^{iuk} \int_0^t \left[\frac{1}{(k-1)!} \lambda^k e^{-\lambda s} s^{k-1} - \frac{1}{(k)!} \lambda^{k+1} e^{-\lambda s} s^k \right] ds &= \\ e^{-\lambda t} + \int_0^t [\lambda e^{-\lambda s} + \lambda e^{\lambda s(\exp(iu)-1)} (\exp(iu) - 1)] ds &= \\ e^{-\lambda t} + \int_0^t \partial_s (-e^{-\lambda s}) ds + \int_0^t \partial_s (e^{\lambda s(\exp(iu)-1)}) ds &= \\ e^{-\lambda t} - e^{-\lambda t} + 1 + e^{\lambda t(\exp(iu)-1)} - 1 &= \\ e^{\lambda t(\exp(iu)-1)} \quad (\text{sic!!}) & \end{aligned}$$

6.20. For each n suppose that X_n has the uniform distribution over the interval $[-n, n]$.

a)

$$\begin{aligned} \phi_{X_n}(t) &= \frac{1}{2n} \int_{-n}^n e^{itx} dx \\ &= \frac{1}{it2n} (e^{int} - e^{-int}) \\ &= \frac{\sin(nt)}{nt} \end{aligned}$$

b) $\phi_{X_n}(0) = 1$ for all n . If $t \neq 0$, $|\phi_{X_n}(t)| \leq \frac{2}{nt}$ and therefore $\phi_{X_n}(t)$ tends to 0.

Note: this is an example where $\lim_{n \rightarrow \infty} \phi_{X_n}(t)$ is not continuous at zero, so the hypothesis of the Levy continuity theorem fails (theorem 6.17) and in fact you can see that the corresponding distributions do not have a limit either.

6.24. Suppose that $\phi(t) = 3\frac{\sin(t)}{t^3} - 3\frac{\cos(t)}{t^2}$ ($t \neq 0$). Actually the limit of this expression at $t = 0$ makes sense and equals 1.

- a) If $\phi = \phi_X$, then X must be symmetric, since ϕ is real.
 b) Using Taylor expansion of sin and cos, we get:

$$\begin{aligned} \phi(t) &= \frac{3}{t^3}(\sin(t) - t \cos(t)) \\ &= \frac{3}{t^3}(t - t^3/3! + t^5/5! \dots - t(1 - t^2/2! + t^4/4! \dots)) \\ &= \frac{3}{t^3}(t^3(1/2! - 1/3!) - t^5(1/4! - 1/5!) + \dots) \\ &= 3(1/3 - t^2(1/4! - 1/5!) + t^4(1/6! - 1/7! \dots)) \end{aligned}$$

The typical coefficient of t^n for n equals up to sign

$$3\frac{1}{n!} \frac{1}{(n+1)(n+2)} \left(1 - \frac{1}{n+3}\right) = 3\frac{1}{n!} \frac{1}{(n+1)(n+3)}$$

Therefore, for $n \geq 1$, $E(X^{2n}) = \frac{3}{(2n+1)(2n+3)}$.

- c) To see that $P(|X| > 1) = 0$, we show that is less than any positive number. Using Chebyshev's inequality, we get: $P(|X| > 1) \leq \frac{E(X^{2n})}{1^{2n}} = \frac{3}{(2n+1)(2n+3)}$. Let $n \rightarrow \infty$, we get that $P(|X| > 1) = 0$.
 d) To see that X is absolutely continuous, we note that $\int_{-\infty}^{+\infty} |\phi(t)| dt < \infty$. Reason: From the Taylor series expansion we can see that it is continuous every where. Since $|\phi(t)| = O(\frac{1}{t^2})$ for large t , this integral is convergent. Now apply Theorem 6.7.

Note: It is a different problem altogether to determine if there is a random variable X for which $\phi = \phi_X$.