AMS 569 - Homework #6

Section 8.8

8.7. To say that $E(X|Y) = Y$ implies (think orthogonal projection) that $E((X-Y)Y) = < X-Y, Y >$ $= 0$. Since also $E(Y|X) = X$ also then $< X-Y, X >= 0$. But then $Var(X-Y) = < X-Y, X-Y >= 0$. But then X and Y are equal almost surely.

8.9. Prove that for bounded random variables X and Z: $E(E(X|Y_1, ..., Y_n)Z) = E(E(Z|Y_1, ..., Y_n)X)$. Let V be the set of funtions in L^2 which are functions of $(Y_1, ..., Y_n)$. Then $X - E(X|Y_1, ..., Y_n)$ and $Z - E(Z|Y_1, ..., Y_n)$ are orthogonal to V. Now $E(E(X|Y_1, ..., Y_n)Z) = E(E(X|Y_1, ..., Y_n)(Z - E(Z|Y_1, ..., Y_n)) + E(E(X|Y_1, ..., Y_n)(E(Z|Y_1, ..., Y_n)) = E(E(X|Y_1, ..., Y_n)(E(Z|Y_1, ..., Y_n))$, because $E(X|Y_1, ..., Y_n) \in V$. Since the right hand side is symmetric in X and Z, we must have:

$$E(E(X|Y_1, ..., Y_n)Z) = E(E(Z|Y_1, ..., Y_n)X)$$

8.12. We will take a short cut in the solution, since we are also doing problem 8.18. If we do 8.18 in full generality,we will get that if (X, Y) are bivariate normal, then $E(X|Y) = a + bY$, where a and b are certain constants, determined as follows: $X - a - bY$ is orthogonal to both the constants and to Y that is:

$$0 = E(X - a - bY) = E(X) - a - bE(Y) \quad (1)$$
$$0 = E(XY - aY - bYY) = E(XY) - aE(Y) - bE(Y^2) \quad (2)$$

i.e.

$$a + bE(Y) = E(X)aE(Y) + bE(Y^2) = E(XY) \quad (3)$$

we get as solution:

$$a = \frac{E(X)E(Y^2) - E(XY)E(Y)}{E(Y^2) - E(Y)^2} \quad (4)$$
$$b = \frac{E(X)E(Y) - E(XY)}{E(Y)^2 - E(Y^2)} \quad (5)$$

Expressing $E(XY)$ in terms of correlation ρ gives result.

8.17. Let $X_1, X_2, ...$ be iid with $E(X_i) = \mu$, and $\sigma^2 = Var(X_i) < \infty$. Define the partial sums $S_n = \sum_i^n X_i$, with $S_0 = 0$. Let N be positive, integer-valued and independent of the sequence X_n. Consider the random sum $S_N = \sum_1^N X_i$.

a. Show that $E(S_N|N) = \mu N$ is easy, since on the left hand side N is though of as constant. Since $E(S_N) = E(E(S_N|N))$, we obtain that $E(S_N) = E(\mu N) = \mu E(N)$.

b. $E(S_N^2|N) = E(S_N^2 - E(S_N)^2|N) + E(E(S_N)^2|N) = N\sigma^2 + (N\mu)^2$. Therefore $E(S_N^2) = E(E(S_N^2|N)) = \sigma^2 E(N) + \mu^2 E(N^2)$. Reshuffeling yields then that:

$$Var(S_N) = E(S_N^2 - E(S_N)^2) = \sigma^2 E(N)^+ \mu^2 Var(N)$$

c. Since the X_i are independent, we get that $E(e^{itS_N}|N) = \Pi_i^n \phi_{X_i}(t) = \phi_{X_1}(t)^n$ since they are identically distributed. Therefore $\phi_{S_N}(t) = E(e^{itS_N}) = E(E(e^{itS_N}|N) = E(\phi_{X_1}(t)^N)$. From definition 6.30, we have that $\zeta_N(u) = E(u^N)$, and therefore $\phi_{S_N}(t) = \zeta_N(\phi_{X_1}(t))$

8.18 (assume $\mu = 0$). To say that $(X, Y_1, ..., Y_n)$ has a multivariate normal distribution with mean zero is to say that the pdf for $(X, Y_1, ..., Y_n)$ is of the form (see section 4.5.5):

$$p(x, y_1, ..., y_n) = C\, exp^{-1/2(a_0^{-2}(x - \sum_i (a_i y_i))^2 + \text{quadratic in } y_i)}$$

Here C and $a_0, ...a_n$ are constants. We can compute $E(X|Y_1 = y_1, ..., Y_n = y_n)$ as $\frac{\int_{-\infty}^{\infty} xp(x,y_1,...,y_n)dx}{\int_{-\infty}^{\infty} p(x,y_1,...,y_n)dx}$ Clearly the term quadratic in y does not matter, nor does C. Therefore we are left with the computation fo the mean, and get $\sum_i (a_i y_i)$ as result. this is a linear function in $y_1, ..., y_n$.
Note: without the hypothesis that $\mu = 0$ we would get also a constant term in the conditional expectation.

8.20. X has $P(\lambda)$ distribution and conditional on X, Y has $N(0, X)$ distribution.

a. the characteristic function for Y is: $\phi_Y(t) = E(e^{itY}) = E(E(e^{itY}|X)) = E(e^{-xt^2/2}) = \sum_0^{\infty} e^{-kt^2/2} e^{\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_0^{\infty} \frac{(e^{-t^2/2}\lambda)^k}{k!} = e^{-\lambda + \lambda exp(-t^2/2)}$. So:

$$\phi_Y(t) = e^{-\lambda + \lambda exp(-t^2/2)}$$

b. In order to see that Y is not absolutely continuous it suffices to note that as $t \to \pm\infty$ $\phi_Y(t) \to e^{-\lambda}$ which is not equal to zero. therefore $\int_{-\infty}^{+\infty} |\phi_Y(t)|dt = \infty$.

c. Let $Z = \frac{Y}{\sqrt{\lambda}}$. Then $\phi_Z(t) = \phi_Y(t/\sqrt{\lambda}) = e^{\lambda(exp(-\frac{t^2}{2\lambda})-1)}$. As $\lambda \to \infty$, the right hand side converges to $e^{-\frac{t^2}{2}}$ and therefore $Z \to N(0, 1)$.

9.9. Let $Y_1, Y_2, ...$ be independent, suppose that $E(Y_k) = 0$ and that $s^2 = Var(Y_k) < \infty$. Let $S_n = \sum Y_k$ and $\sigma_n^2 = \sum_1^n s_k^2$. Prove that $X_n = S_n^2 - \sigma_n^2$ is a martingale.
$E(X_{n+1}|Y_1, ..., Y_n) = E(S_{n+1}^2|Y_1, ..., Y_n) - \sigma_{n+1}^2) = E(Y_{n+1}^2|Y_1, ..., Y_n) + 2E(Y_{n+1}(Y_1 + ...Y_n)|Y_1, ...Y_n) + S_n^2 = E(Y_{n+1}^2) + 2E(Y_{n+1}|Y_1, ...Y_n)(Y_1 + ... + Y_n) + S_n^2 - \sigma_{n+1}^2 = s_{n+1}^2 + 0 + S_n^2 - \sigma_{n+1}^2 = X_n$

9.10. This problem has a fair number of words in it, but there is less to it. Let $Y_1, ..., Y_n$ be the outcomes of the first n games. Let's say X_n is the gambler's net gain after n games. Let $B(Y_1, ..., Y_n)$ be the gambler's next bet. Then with probability .5 (case he wins) his winning will be $X_{n+1} = X_n + B(Y_1, ..., Y_n)$, and with probabiliy .5 (case he looses) his winning will be $X_{n+1} = X_n - B(Y_1, ..., Y_n)$ we have formula: $X_{n+1} = X_n + Y_{n+1}B(Y_1, ..., Y_n)$. Anyway: $E(X_{n+1}|Y_1, ..., Y_n) = X_n$, so X_n is a martingale, regardless of the details of the betting scheme $B(Y_1, ..., Y_n)$.