AMS 210  Practice Final A       Prof Tucker

1. Consider the following growth model $S' = 1.2S - .4T$, $T' = .2S + .6T$
   a) Find the eigenvalues and vectors for the associated matrix $A$.
   b) Write $A$ in the form $UDU^{-1}$

2. Given $Ax = b$, where $A = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & 1 \\ 1 & -5 & 5 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} -5 & -5 & 1 \\ -3 & -5/2 & 1/2 \\ -2 & -3/2 & 1/2 \end{bmatrix}$
   a) Find $x$ when $b = [100, 100, 100]$.
   b) If $b_2$, the second component of right-side vector $b$, is increased by 10, how will $x_1$, the first component in the solution $x$, change?
   c) Give the LU decomposition of $A$ and write $A$ as a sum of simple matrices.
   d) What is the determinant of $A$? Explain how you found it from part c).

3. Find the steady vector $p$, such that $Ap = p$ for the following Markov chain matrix

\[
A = \begin{bmatrix}
1/2 & 1/2 & 0 & 0 & 0 \\
1/2 & 1/4 & 1/2 & 0 & 0 \\
0 & 1/4 & 1/4 & 1/2 & 0 \\
0 & 0 & 1/4 & 1/4 & 1/2 \\
0 & 0 & 0 & 1/4 & 1/2
\end{bmatrix}
\]

4. a) Find the condition number of $A = \begin{bmatrix} 5 & -2 \\ -6 & 3 \end{bmatrix}$ using the sum norm.
   b) If the 5 in $A$ is changed to 4 to get $A'$, give bound (using the sum norm) on the ratio $|e|/|x+e|$, where $x$ is the solution to $Ax = b$ for some $b$, and $x+e$ is the solution to $A'(x+e) = b$.

5. Suppose that the first and fifth state in the Markov chain in Question #3 are made into absorbing states.
   If $Q$ is the submatrix of non-absorbing states (states 2,3,4), then $N=(I-Q)^{-1}=\begin{bmatrix} 28 & 24 & 16 \\ 12 & 36 & 24 \\ 4 & 12 & 28 \end{bmatrix}$
   If you start in the middle state (state 3 in the original Markov chain),
   a) What is the expected number of times you visit state 2?
   b) What is the expected number of rounds before you are absorbed in states 1 or 5
   c) What is the probability of being absorbed in state 5?
6. Supply the following information about $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -1 \\ -3 & -1 & 1 & 8 \\ -1 & 1 & 1 & 6 \end{bmatrix}$

a) The basis for the range of $\mathbf{A}$.

b) The basis for the null space of $\mathbf{A}$.

c) Constraint(s) on vectors in the range of $\mathbf{A}$.

d) The rank of $\mathbf{A}$.

7. Which of the following properties guarantees that the 4-by-4 matrix $\mathbf{A}$ is invertible, which guarantee that $\mathbf{A}$ is not invertible (possibly, some may guarantee neither).

a) The range of $\mathbf{A}$ has dimension 4.

b) The determinant of $\mathbf{A}$ equals 0.

c) The null space of $\mathbf{A}^T$ has dimension 1.

d) The rows of $\mathbf{A}$ are linearly independent.

e) The columns of $\mathbf{A}$ are linearly dependent.

8. In a medical experiment, levels of Drug A and Drug B are set and the level of protein P in the blood is measured. The data is:

<table>
<thead>
<tr>
<th>Protein A</th>
<th>Drug A</th>
<th>Drug B</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>-3</td>
</tr>
</tbody>
</table>

a) For the regression model $P = qA + rB + s1$, find the pseudoinverse of the coefficient matrix $\mathbf{A}$ for $q, r$ and $s$ (recall that $s$'s column is all 1's), and then find the values of the regression coefficients $q, r,$ and $s$ (Hint: columns are orthogonal on the right side of the regression equation).

b) What is the correlation coefficient between drug A and drug B?
1. Consider the following growth model \( S' = 1.3S - .2T, T' = .15S + .9T \)
   a) Find the eigenvalues and vectors for the associated matrix \( A \).
   b) Write \( A \) in the form \( \text{UDU}^{-1} \).

2. Given \( Ax = b \) where \( A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 1 & 4 & 3 \end{bmatrix} \) and \( A^{-1} = \begin{bmatrix} 5/2 & -1 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ -3/2 & 1 & -1/2 \end{bmatrix} \)
   a) Find \( x \) when \( b = [200, 200, 200] \).
   b) If \( b_1 \), the first component of right-side vector \( b \), is increased by 1, how will \( x_3 \), the third component in the solution \( x \), change?
   c) Give the LU decomposition of \( A \) and write \( A \) as a sum of simple matrices.
   d) What is the determinant of \( A \)? Explain how you found it from partc).

3. Find the steady vector \( p \), such that \( Ap = p \) for the following Markov chain matrix
   \[
   A = \begin{bmatrix}
   1/3 & 1/3 & 0 & 0 & 0 \\
   2/3 & 1/2 & 1/3 & 0 & 0 \\
   0 & 1/6 & 1/2 & 1/3 & 0 \\
   0 & 0 & 1/6 & 1/2 & 2/3 \\
   0 & 0 & 0 & 1/6 & 1/3 \\
   \end{bmatrix}
   \]
   a) Find the condition number of \( A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \) using the sum norm.
   b) If the 3 in \( A \) is changed to 2 to get \( A' \), give bound (using the sum norm) on the ratio \( |e|/|x+e| \), where \( x \) is the solution to \( Ax = b \) for some \( b \), and \( x+e \) is the solution to \( A'(x+e) = b \).

4. Suppose that the first and fifth state in the Markov chain in Question #3 are made into absorbing states.
   If \( Q \) is the submatrix of non-absorbing states (states 2,3,4), then \( N=(I-Q)^{-1}=1/5 \begin{bmatrix} 14 & 12 & 8 \\ 6 & 18 & 12 \\ 2 & 6 & 14 \end{bmatrix} \)
   If you start in the middle state (state 3 in the original Markov chain),
   a) What is the expected number of times you visit state 4?
   b) What is the expected number of rounds before you are absorbed in states 1 or 5?
   c) What is the probability of being absorbed in state 1?
6. Supply the following information about \( A = \begin{bmatrix} 2 & 0 & 1 & 6 \\ 2 & 4 & 3 & 2 \\ 1 & 3 & 2 & 0 \end{bmatrix} \)

a) Columns generating the range of \( A \).
b) Vector(s) generating the null space of \( A \).
c) Constraint(s) on vectors in the range of \( A \).
d) The rank of \( A \).

7. Which of the following properties guarantees that the \( n \)-by-\( n \) matrix \( A \) is invertible, which guarantee that \( A \) is not invertible (possibly, some may guarantee neither).

a) \( \text{Rank}(A) = n \).
b) The dimension of the column space equals the dimension of the row space.
c) The null space of \( A^T \) has dimension 0.
d) The columns of \( A \) are linearly dependent.
e) The matrix \( A^T A \) is invertible.

8. A statistical experiment is run on a machine that makes paper bags. The settings of dial A and dial B effect the quality of the bags. We use the regression model \( z = qx + ry + s \), where \( z \) is the quality of a bag, \( x \) is dial A's setting, \( y \) is dial B's setting. The results of the experiment are

\[
\begin{array}{ccc}
2 & x & y \\
4 & 0 & 0 \\
5 & 4 & 1 \\
9 & -2 & -1 \\
8 & 0 & -3 \\
4 & -2 & 3 \\
\end{array}
\]

a) For the regression model \( P = qA + rB + s1 \), find the pseudoinverse of coefficient matrix \( A \) for \( q, r \) and \( s \) (recall that \( s \)'s column is all 1's), and then find the values of the regression coefficients \( q, r, \) and \( s \) (Hint: columns are orthogonal on the right side of the regression equation).
b) What is the correlation coefficient between \( x \) and \( z \) (note: these two columns are NOT orthogonal)?