23. Give a constraint equation, if one exists, on the probability vectors in the range of the transition matrices in Exercise 15.

24. By looking at the transition matrix for the frog Markov chain, explain why the even-state probabilities in the next period must equal the odd-state probabilities in the next period.

*Hint:* Show that this equality is true if we start (this period) from a specific state.

25. Compute the constraint on the range of the following powers of the frog transition matrix. You will need a matrix software package.
   (a) $A^7$  
   (b) $A^3$  
   (c) $A^{10}$

26. Show that the range $R(A)$ of a matrix is a vector space. That is, if $b$ and $b'$ are in $R(A)$ (for some $x$ and $x'$, $Ax = b$ and $Ax' = b'$), show that $rb + sb'$ is in $R(A)$, for any scalars $r$, $s$.

27. Using matrix algebra, show that if $x_1$ and $x_2$ are solutions to the matrix equation $Ax = b$, then any linear combination $x' = cx_1 + dx_2$, with $c + d = 1$, is also a solution.

28. Show that the intersection $V_1 \cap V_2$ of two vector spaces $V_1$, $V_2$ is again a vector space.

---

### Section 5.2 Theory of Vector Spaces Associated with Systems of Equations

In this section we introduce basic concepts about vector spaces and use them to obtain important information about the range and null space of a matrix. Recall that a vector space $V$ is a collection of vectors such that if $u$, $v \in V$, then any linear combination $ru + sv$ is in $V$. In Section 5.1 we introduced the range and null space of a matrix $A$:

$$\text{Range}(A) = \{b : Ax = b \text{ for some } x\}$$

$$\text{Null}(A) = \{x : Ax = 0\}$$

We noted that $\text{Range}(A)$ and $\text{Null}(A)$ are both vector spaces.

In Examples 2, 3, and 4 of Section 5.1 we used the elimination process to find a vector or pair of vectors that generated the null spaces of certain matrices. For example, multiples of $[-1, 2, -2, 2, -2, 1]$ formed the null space of the frog Markov transition matrix. In Examples 7, 8, and 9 of Section 5.1, we used elimination to find constraint equations that vectors in the range must satisfy. For the frog Markov matrix, the constraint for range vectors $p$ was $p_1 + p_3 + p_5 = p_2 + p_4 + p_6$. 

The number of vectors generating the null space and number of constraint equations for the range were dependent on how many pivots we made during elimination. Our goal in this section is to show that the sizes of Null(A) and Range(A) are independent of how elimination is performed.

The vector space \( V \) generated by a set \( Q = \{q_1, q_2, \ldots, q_k\} \) of vectors is the collection of all vectors that can be expressed as a linear combination of the \( q_i \)'s. That is,

\[ V = \{v : v = r_1q_1 + r_2q_2 + \cdots + r_kq_k, \ r_i \text{ scalars}\} \]

For example, if \( Q \) consists of the unit \( n \)-vectors \( e_j \) (with all 0's except for a 1 in position \( j \)), then \( V \) is the vector space of all \( n \)-vectors, that is, euclidean \( n \)-space. Another name for a generating set is a spanning set.

The column space of \( A \), denoted \( \text{Col} (A) \), is the vector space generated by the column vectors \( a_j^C \) of \( A \). When we write \( Ax = b \) as

\[ a_1^C x_1 + a_2^C x_2 + \cdots + a_n^C x_n = b \]

we see that the system \( Ax = b \) has a solution if and only if \( b \) can be expressed as a linear combination of the column vectors of \( A \), or

**Lemma 1.** The system \( Ax = b \) has a solution if and only if \( b \) is in \( \text{Col}(A) \). Equivalently, \( \text{Col}(A) = \text{Range}(A) \).

The components \( x_j \) of the solution \( x \) give the weights in the linear combination of columns that yield \( b \). Note that Lemma 1 is true for any \( m \)-by-\( n \) matrix \( A \) and any \( m \)-vector \( b \).

**Example 1. Refinery Problem as a Column Space Problem**

The refinery problem introduced in Section 1.2 involved three refineries each producing different amounts of heating oil, diesel oil, and gasoline from a barrel of crude oil. Production levels of each refinery were sought to satisfy a vector of demands. The resulting system of equations was

- Heating oil: \( 20x_1 + 4x_2 + 4x_3 = 500 \)
- Diesel oil: \( 10x_1 + 14x_2 + 5x_3 = 850 \)
- Gasoline: \( 5x_1 + 5x_2 + 12x_3 = 1000 \)

But this system is just seeking to express the demand vector \( [500, 850, 1000] \) as a linear combination of the production vectors of the three refineries. That is, we seek \( x_1, x_2, x_3 \) such that:

\[
x_1 \begin{bmatrix} 20 \\ 10 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 14 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 5 \\ 12 \end{bmatrix} = \begin{bmatrix} 500 \\ 850 \\ 1000 \end{bmatrix}
\]
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Up to this point, solving a system $Ax = b$ was viewed as a problem about the rows of $A$, that is, about the equations specified by the rows. Gaussian elimination involves forming linear combinations of the equations (rows of $A$) to obtain a new reduced system that can be solved by back substitution. Lemma 1 says that solving $Ax = b$ can equally be viewed as a problem about a linear combination of the columns of $A$. This vector approach to solving $Ax = b$ has an associated geometric picture.

Example 2. Geometric Picture of Solution to a System of Equations

Consider the system of equations

$$
\begin{align*}
    x_1 + x_2 &= 4 \\
    x_1 - 2x_2 &= 1
\end{align*}
$$

or

$$
\begin{bmatrix}
    1 & 1 \\
    1 & -2
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix}
=
\begin{bmatrix}
    4 \\
    1
\end{bmatrix}
$$

Solving by elimination, we find that $x_1 = 3$ and $x_2 = 1$. Figure 5.3 graphs this solution in vector-space terms, showing the right-side vector $[4, 1]$ as a linear combination of the column vectors $[1, 1]$ and $[1, -2]$. Note that the picture gives no insight into why $x_1 = 3$, $x_2 = 1$ is the solution.

To determine the size of Range($A$), we analyze the structure of Col($A$), the column space of $A$, which by Lemma 1 equals Range($A$). The key question is: How many of the columns of $A$ are actually needed to generate Col($A$). Some columns in $A$ may be redundant.

A set of vectors $a_1, a_2, \ldots, a_r$ is called linearly dependent if one of them can be expressed as a linear combination of the others. Another way to say this is that there is a nonzero solution $x$ to

$$
x_1 a_1 + x_2 a_2 + \cdots + x_r a_r = 0 \quad \text{or, equivalently,} \quad Ax = 0 \quad (1)
$$

Figure 5.3
where $A$ is the matrix whose columns are the vectors $a_i$. Linear dependence is equivalent to (1) because if $x_i \neq 0$, we can rewrite (1) as

$$-x_i a_i = x_1 a_1 + \cdots + x_n a_n$$

or

$$a_i = -\frac{x_1}{x_i} a_1 - \frac{x_2}{x_i} a_2 \cdots - \frac{x_n}{x_i} a_n$$

So any $a_i$, for which $x_i \neq 0$, can be written as a linear combination of the other $a$'s.

A set of vectors are **linearly independent** if they are not linearly dependent. For example, the columns of an identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are linearly independent. If vectors $a_i$ are linearly independent, then the only solution to $x_1 a_1 + \cdots + x_n a_n = 0$ (i.e., $Ax = 0$) can be $x = 0$.

### Example 3. Example of a Linearly Dependent Set of Columns

Consider the matrix $A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & -2 & 1 \end{bmatrix}$. By inspection we see that

$$a_3 = 3a_1 + a_2: \quad \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

So the columns of $A$ are linearly dependent.

The following method illustrates a systematic way to find this linear dependence. We perform elimination by pivoting on $A$. We pivot on entry $(1, 1)$ and then on $(2, 2)$:

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & -3 & -3 \end{bmatrix} \rightarrow A^* = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

where $A^*$ represents the reduced form of $A$.

Remember that $x$ is a solution to $Ax = 0$ if and only if $x$ is a solution to $A^*x = 0$. Equivalently, there is a linear dependence among the columns of $A$ if and only if there are is linear dependence among the columns of $A^*$. But the first two columns of $A^*$ are unit vectors (they form the 2-by-2 identity matrix). So trivially, the third column of $A^*$ is dependent on the first two:
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\[
\mathbf{a}_3^* = 3\mathbf{a}_1^* + \mathbf{a}_2^*:
\begin{bmatrix}
3 \\
1
\end{bmatrix} = 3 \begin{bmatrix}
1 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

(3)

This is exactly the relationship that we found in (2).

If we were to apply the method in Example 3 to the coefficient matrix in the refinery problem of Example 1, elimination by pivoting would reduce the matrix to a 3-by-3 identity matrix. Since the columns of the identity matrix are trivially linearly independent, this means that the original columns in the refinery problem were linearly independent.

We state the method used to find linear dependence in Example 3 and its consequences as a theorem.

**Theorem 1**

(i) Let \( \mathbf{A} \) be any \( m \times n \) matrix and let \( \mathbf{A}^* \) be the reduced matrix obtained from \( \mathbf{A} \) using elimination by pivoting. Then a set of columns of \( \mathbf{A} \) is linearly dependent (linearly independent) if and only if the corresponding columns in \( \mathbf{A}^* \) are linearly dependent (linearly independent).

(ii) Any unpivoted column of \( \mathbf{A}^* \) (a column that was not reduced to a unit vector) is linearly dependent on the set of columns containing pivots.

(iii) The columns of \( \mathbf{A}^* \) with pivots are linearly independent. The corresponding columns of \( \mathbf{A} \) generate the column space of \( \mathbf{A} \).

The following example illustrates this method further.

**Example 4. Redundant Columns in Transportation Problem Constraints**

In Example 4 of Section 5.1 we examined the following system of equations (that were transportation problem constraints seen in Section 4.6).

\[
\begin{align*}
\mathbf{x}_1 + \mathbf{x}_2 & = 20 \\
\mathbf{x}_3 + \mathbf{x}_4 & = 30 \\
\mathbf{x}_5 + \mathbf{x}_6 & = 15 \\
\mathbf{x}_1 + \mathbf{x}_3 + \mathbf{x}_5 & = 25 \\
\mathbf{x}_2 + \mathbf{x}_4 + \mathbf{x}_6 & = 40
\end{align*}
\]

with coefficient matrix \( \mathbf{A} \)

\[
\mathbf{A} = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
\]

(4)
In Section 5.1 we performed elimination by pivoting on A using entries (1, 2), (2, 4), (3, 6) and (4, 3). The reduced matrix \( A^* \) was

\[
A^* = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(5)

First note that the last row of zeros can be ignored. Columns 2, 4, 6, and 3 of \( A^* \) (in that order) are the unit vectors of the 4-by-4 identity matrix. Columns 1 and 5 of \( A^* \) are each linearly dependent on the four unit-vector columns. For example,

\[
a_1^* = a_2^* - a_4^* + a_5^*:
\begin{bmatrix}
1 \\
-1 \\
0 \\
1 \\
0 \\
\end{bmatrix}
= \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
- \begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
0 \\
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

(6)

The relation (6) among columns in \( A^* \) is mirrored in A, where

\[
a_1 = a_2 - a_4 + a_5:
\begin{bmatrix}
1 \\
0 \\
1 \\
0 \\
0 \\
\end{bmatrix}
= \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
- \begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
0 \\
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

Thus the columns where pivots were performed, columns \( a_2, a_3, a_4, \) and \( a_6 \), generate the range of A.

From Theorem 1, part (iii), it follows that the number of pivots performed equals the number of linearly independent columns that generate the column space of A.

A basis of a vector space \( V \) is a minimal-sized set of vectors that generate \( V \). Implicit in this definition is the fact that a basis is a set of linearly independent vectors. As an example, the \( n \) coordinate vectors \( e_j \) form a basis for the space of all \( n \)-dimensional vectors. Since a basis is a minimal-sized generating set, every generating set contains a basis. For example, while the column space of a matrix \( A \) is defined to be generated by the columns of \( A \), only the pivot columns are needed to generate the column space, as shown in Example 4.

The following result, which we prove in two ways, shows the theo-
retical relationship among the concepts of basis, linear independence, and unique solution.

**Proposition.** If \( \{v_1, v_2, \ldots, v_n\} \) is a basis for a vector space \( V \), every vector in \( V \) has a unique representation as a linear combination of the \( v_i \)'s.

**Proof 1 (Using the Definition of Linear Independence):** Suppose that \( w \) is a vector in \( V \) that has two representations as a linear combination of the \( v_i \). So
\[
\begin{align*}
  w &= a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \\
  w &= b_1 v_1 + b_2 v_2 + \cdots + b_n v_n
\end{align*}
\]

Then
\[
0 = w - w = (a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_n - b_n)v_n
\]

(7)

Since the \( v_i \)'s form a basis and hence are linearly independent, the linear combination of \( v_i \)'s in (7) can only equal \( 0 \) if the terms \( a_i - b_i \) are all zero. So the two representations must be the same. \( \blacksquare \)

**Proof 2 (Using Elimination).** To find the representation of \( w \) in terms of the \( v_i \)'s, we solve the system of equations for the \( x_i \)'s:
\[
x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = w \quad \text{or} \quad Ax = w
\]

(8)

where \( A \) has the \( v_i \)'s as its columns. Since the \( v_i \)'s are a basis and hence linearly independent, we can pivot in every column [otherwise, by Theorem 1, part (ii), each unpivoted column is linearly dependent on the pivot columns]. Then \( Ax = w \) has a unique solution (see the summary at the end of Section 5.1). \( \blacksquare \)

Proof 2 shows us how to compute the unique representation of a vector \( w \) in terms of the \( v_i \)'s, simply solve (8). For example, if \( v_1 = [1, 2, 3] \) and \( v_2 = [0, -1, 2] \) are a basis for vector space \( V \) and \( w = [3, 8, 5] \) is in \( V \), then to determine the right linear combination of the \( v_i \)'s to get \( w \), we solve the system

\[
\begin{bmatrix}
1 & 0 \\
2 & -1 \\
3 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
3 \\
8 \\
5
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
3 \\
-2 \\
0
\end{bmatrix}
\]

(9)

or \( x_1 = 3, x_2 = -2 \).
Now we prove a critical vector-space lemma that resolves the question of whether all pivots sequences have the same length.

**Lemma 2.** All linearly independent sets of vectors that generate a given vector space \( V \) have the same size. Any such set is a basis for \( V \).

**Proof.** Let \( S = \{ s_i \} \) and \( T = \{ t_i \} \) be two sets of linearly independent vectors that generate \( V \). Suppose that \( S \) and \( T \) have different sizes; for concreteness, let \( S \) have four vectors and \( T \) have five vectors. Then we can use linear combinations of the vectors of \( S \) to represent the vectors in \( T \). If \( t_1 = c_1s_1 + c_2s_2 + c_3s_3 + c_4s_4 \), define the \( S \)-coordinate vector of \( t_1 \) to be \( [c_1, c_2, c_3, c_4] \). Consider the equation, defined in (1), for dependence of the \( t_i \) (with the \( t_i \) represented in \( S \)-coordinates):

\[
x_1t_1 + x_2t_2 + x_3t_3 + x_4t_4 + x_5t_5 = 0
\]

Since the \( t_i \) are four-dimensional vectors (in \( S \)-coordinates), (10) is a system with four equations in five variables. Solving (10) by elimination by pivoting leaves at least one unpivoted column and hence by Theorem 1, part (ii), there is linear dependence among the \( S \)-coordinate vectors. \( \blacksquare \)

The **dimension** of a vector space \( V \), written \( \dim(V) \), is the number of vectors in a basis for \( V \). For example, the set of \( n \) unit vectors \( e_i \) is a basis for \("n-dimensional space"\); thus this space does indeed have dimension \( n \).

Combining Lemma 2 with Theorem 1, part (ii), we have

**Theorem 2**

(i) The columns of \( A \) used in a pivot sequence are a basis for the range of \( A \).

(ii) All pivot sequences have the same size; the size is the dimension of the range of \( A \).

By Theorem 2, it now makes sense to talk about the number of pivots in a pivot sequence. The **rank** of a matrix \( A \), written \( \text{rank}(A) \), is the number of pivots in any pivot sequence.

**Corollary**

\[
\text{Rank}(A) = \dim(\text{Range}(A))
\]

We have been concerned about which sets of columns of \( A \) are linearly dependent, that is, when there is a nonzero \( x \) so that

\[
x_1a_1 + x_2a_2 + \cdots + x_na_n = 0 \quad \text{or} \quad Ax = 0
\]

Such an \( x \) in (11) is a vector in \( \text{Null}(A) \), the null space of \( A \).

If \( A^* \) is the reduced matrix, then we know that \( x \) is a solution of
Sec. 5.2 Theory of Vector Spaces Associated with Systems of Equations

\[ Ax = 0 \] if and only if it is a solution to \( A^*x = 0 \). Thus \( \text{Null}(A^*) = \text{Null}(A) \).

**Example 5. Relation Between Null Space and Column Space**

Consider the matrix

\[
A = \begin{bmatrix}
1 & 2 & 3 & 1 \\
1 & -2 & -1 & -3 \\
-2 & 3 & 1 & 5
\end{bmatrix}
\]

and perform elimination by pivoting.

Pivoting on entry (1, 1) yields

\[
\begin{bmatrix}
1 & 2 & 3 & 1 \\
0 & -4 & -4 & -4 \\
0 & 7 & 7 & 7
\end{bmatrix}
\]

Pivoting on entry (2, 2) yields

\[
A^* = \begin{bmatrix}
1 & 0 & i & -1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Clearly, the first two columns of \( A^* \), the pivot columns, are a basis for the column space of \( A^* \)—they are linearly independent and generate \( \text{Col}(A^*) \). Then by Theorem 1, part (iii), the first two columns in \( A \) generate \( \text{Col}(A) \).

Since \( \text{Null}(A^*) = \text{Null}(A) \), a basis for \( \text{Null}(A^*) \) will be a basis for \( \text{Null}(A) \). Looking at \( A^* \), we see that

\[
a_3^* = a_1^* + a_2^* \quad \text{and} \quad a_4^* = -a_1^* + a_2^*
\]

(where \( a_i^* \) denotes the \( i \)-th column of \( A^* \)). The vector equations in (12) can be rewritten as

\[
-a_1^* - a_2^* + a_3^* = 0 \quad \text{or} \quad A^* \begin{bmatrix}
-1 \\
1 \\
1
\end{bmatrix} = 0
\]

\[
a_1^* - a_2^* + a_4^* = 0 \quad \text{or} \quad A^* \begin{bmatrix}
-1 \\
0 \\
1
\end{bmatrix} = 0
\]
Let \( x_3^* = [-1, -1, 1, 0] \) and \( x_4^* = [1, -1, 0, 1] \). Since \( x_3^* \) and \( x_4^* \) are linearly independent (look at their last two entries) and generate \( \text{Null}(A^*) \), they form a basis for \( \text{Null}(A^*) \) and hence for \( \text{Null}(A) \).

In general, given a reduced matrix \( A^* \), each unpivoted column can be expressed as a linear combination of the pivoted columns, which are unit vectors in \( A^* \) [see (12)]. This linear combination yields a solution to \( A^* x = 0 \), as shown in (13). If unpivoted column \( h \) of \( A^* \) has an entry \( a_{ih}^* \) in the \( i \)th row, the solution \( x_h^* \) we obtain is

\[
x_h^* = [-a_{1h}^*, -a_{2h}^*, \ldots, -a_{mh}^*, 0, 0, \ldots, 1, \ldots, 0]
\]

where the entries for unpivoted columns are all 0 except for entry \( h \), which is 1 (assuming pivots were performed in the first \( m \) rows and \( m \) columns). For example, in Example 5, the fourth column of \( A^* \) begins

\[
\begin{bmatrix}
-1 \\
1 \\
\vdots
\end{bmatrix}
\]

(in the two pivot rows), so \( x_4^* = [1, -1, 0, 1] \). The entries in (14) from the unpivoted columns form a unit vector, so the set of \( x_h^* \)'s are linearly independent and form a basis of the null space of \( A \).

Observe that every column in \( A^* \) is now either (i) a unit vector that is in the basis of the column space; or else (ii) gives rise to a vector in the basis of the null space. That is, every column contributes to the size of the range of \( A \) or to the diversity of different solutions possible to \( Ax = b \), for a given \( b \).

**Theorem 3.** Let \( A \) be a matrix with \( n \) columns. The vectors \( x_h^* \) in (14) corresponding to unpivoted columns form a basis for \( \text{Null}(A) \). Furthermore,

\[
\dim(\text{Range}(A)) + \dim(\text{Null}(A)) = n
\]

**Corollary A**

\[
\dim(\text{Null}(A)) = n - \dim(\text{Range}(A)) = n - \text{rank}(A)
\]

**Corollary B.** Any solution \( x' \) to \( Ax = b \) can be written in the form

\[
x' = x^* + r_1 x_1^* + r_2 x_2^* + \cdots + r_k x_k^*
\]

where \( x^* \) is a given particular solution to \( Ax = b \) and the \( x_h^* \)'s are as given in (14).
Proof of Corollary B. By Theorem 1, part (iv) of Section 5.1, any solution \( x' \) can be written \( x' = x^* + x^\circ \), the sum of a particular solution \( x^* \) and some null space vector \( x^\circ \). Since the \( x_i^* \) generate Null(\( A \)), any such \( x^\circ \) is some linear combination of the \( x_i^* \)'s. ■

We complete our brief survey of vector spaces of \( A \) with Row(\( A \)), the vector space generated by the rows of \( A \). As noted when elimination was introduced in Section 3.2, the elimination process repeatedly replaces a row with a linear combination of rows. When rows are zeroed out in the elimination process, they are linearly dependent on the preceding rows in which pivots were performed. Conversely, every nonzero row in \( A^* \) is a pivot row (where a pivot was performed).

Because the submatrix of \( A^* \) formed by the pivot rows and pivot columns is an identity matrix, these pivot rows are linearly independent (see Exercises for details) and will be shown shortly to form a basis for Row(\( A \)). Hence the dimension of the row space equals \( \text{rank}(A) \) (= number of pivots).

**Theorem 4.** Let \( A \) be any \( m \)-by-\( n \) matrix. The maximum number of linearly independent rows in \( A \) and the maximum number of linearly independent columns in \( A \) are equal. Both are \( \text{rank}(A) \). That is,

\[
\text{dim}(\text{Row}(A)) = \text{rank}(A) = \text{dim}(\text{Col}(A))
\]

The results in Theorems 2, 3, and 4 yield several more equivalent conditions for when a system of equations has a unique solution.

**Theorem 5.** Let \( A \) be an \( n \)-by-\( n \) matrix. The system \( Ax = b \) has a unique solution, for any \( b \), if and only if any of the following equivalent conditions are satisfied.

(i) The dimension of Range(\( A \)) is \( n \).
(ii) The column vectors of \( A \) are linearly independent.
(iii) The dimension of Row(\( A \)) is \( n \).
(iv) The row vectors of \( A \) are linearly independent.
(v) The null space of \( A \) has dimension 0 (consists of only the 0 vector).

The following example illustrates the uses of Theorem 5.

**Example 6. Row Space Test for Unique Solution**

Let us consider the following variation of our refinery model introduced in Section 1.2. Suppose that we change the numbers in gasoline production so that the third row is the sum of the first two rows.

\[
\begin{align*}
\text{Heating oil:} & \quad 20x_1 + 4x_2 + 4x_3 = 500 \\
\text{Diesel oil:} & \quad 10x_1 + 14x_2 + 5x_3 = 850 \\
\text{Gasoline:} & \quad 30x_1 + 18x_2 + 9x_3 = 1000
\end{align*}
\]
Once we observe that the last row in the coefficient matrix is the sum (a linear combination) of the first two rows, so that the rows are not linearly independent, then we know by Theorem 5, part (iv), that there will not be a unique solution to this production problem: either no solution or multiple solutions. If we tried to perform elimination, we would only be able to pivot twice (if we could pivot in all three rows, they would have to be linearly independent).

Theorem 4 tells us that if the rows are dependent, the columns also are. However, that column dependence is far from obvious. ■

We now give a theoretical application of Theorems 3 and 4. Suppose that A is m-by-n, where m < n. All the columns cannot be linearly independent, since \( \dim(\text{Col}(A)) = \dim(\text{Row}(A)) \) and there are only m rows, m < n. Therefore, \( \text{rank}(A) \leq m < n \). We conclude from Theorem 3, \( \dim(\text{Null}(A)) = n - \text{rank}(A) = n - \dim(\text{Row}(A)) > 0 \). Then \( \text{Null}(A) \) is infinite and \( Ax = b \) cannot have a unique solution:

**Theorem 6.** If \( A \) is an m-by-n matrix, where m < n, the system \( Ax = b \) can never have a unique solution (either multiple solutions or no solution).

We close this section with a discussion of another way to interpret the elimination process and the rank of a matrix. To do this, we must introduce the concept of a simple matrix.

A **simple matrix** \( K \) is formed by the product \( c \cdot d \) of two vectors \( c \) and \( d \) in which \( c \) is treated as an m-by-1 matrix and \( d \) as a 1-by-n matrix. Thus entry \( k_{ij} \) of \( K \) equals \( a_i b_j \). We refer to this product \( c \cdot d \) as a matrix product of vectors. For example, the following matrix product of vectors yields a simple matrix.

\[
[3, -1] \cdot [1, 2, 3] = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 & 3 \cdot 2 & 3 \cdot 3 \\ -1 \cdot 1 & -1 \cdot 2 & -1 \cdot 3 \end{bmatrix}
\]

(16)

\[
= \begin{bmatrix} 3 & 6 & 9 \\ -1 & -2 & -3 \end{bmatrix}
\]

All rows in a simple matrix are multiples of each other, and similarly for columns. If we pivot on an entry \((i, j)\) in a simple matrix, the elimination computation will convert all other rows to 0's (verification is left as an exercise). This means that simple matrices have rank 1.

Simple matrices will be used extensively in Section 5.5. For now, the property of simple matrices of interest is

**Theorem 7.** Let \( C \) be a m-by-\( r \) matrix with columns \( c_i \) and \( D \) be an \( r \)-by-\( n \) matrix with rows \( d_j \). Then the matrix multiplication \( CD \) can be
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decomposed into a sum of the simple matrices $c_i^* * d_i^*$ of the column vectors of $C$ times the row vectors of $D$.

$$CD = c_1^* * d_1^* + c_2^* * d_2^* + \cdots + c_r^* * d_r^*$$  \hspace{1cm} (17)

One way to verify (17) is using the rules for partitioned matrices, that is, we partition $C$ into $r$ $m$-by-$1$ matrices (the $c_i^*$) and partition $D$ into $r$ $1$-by-$n$ matrices (the $d_i^*$); see Exercise 15 of Section 2.6 for details.

We illustrate this theorem with the following product of two matrices.

**Example 7. Decomposition of Matrix Multiplication into a Sum of Simple Matrices**

Let

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 11 & 12 & 13 \\ 14 & 15 & 16 \\ 17 & 18 & 19 \end{bmatrix}$$

Then

$$CD = \begin{bmatrix} 1 \times 11 + 2 \times 14 + 3 \times 17 \\ 4 \times 11 + 5 \times 14 + 6 \times 17 \\ 1 \times 12 + 2 \times 15 + 3 \times 18 \\ 4 \times 12 + 5 \times 15 + 6 \times 18 \\ 1 \times 13 + 2 \times 16 + 3 \times 19 \\ 4 \times 13 + 5 \times 16 + 6 \times 19 \end{bmatrix}$$  \hspace{1cm} (18a)

$$= \begin{bmatrix} 1 \times 11 \\ 4 \times 11 \\ 1 \times 12 \\ 4 \times 12 \\ 1 \times 13 \\ 4 \times 13 \end{bmatrix} + \begin{bmatrix} 2 \times 14 \\ 5 \times 14 \\ 2 \times 15 \\ 5 \times 15 \\ 2 \times 16 \\ 5 \times 16 \end{bmatrix} + \begin{bmatrix} 3 \times 17 \\ 6 \times 17 \\ 3 \times 18 \\ 6 \times 18 \\ 3 \times 19 \end{bmatrix}$$  \hspace{1cm} (18b)

$$= \begin{bmatrix} 1 \\ 4 \\ 1 \\ 4 \\ 1 \\ 4 \end{bmatrix} * \begin{bmatrix} 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \\ 2 \\ 5 \\ 2 \\ 5 \end{bmatrix} * \begin{bmatrix} 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} * \begin{bmatrix} 17 \\ 18 \\ 19 \end{bmatrix}$$  \hspace{1cm} (18c)

$$= c_1^* * d_1^* + c_2^* * d_2^* + c_3^* * d_3^*$$

The first simple matrix $c_1^* * d_1^*$ in (18c) is a matrix containing the first term of each scalar product in the entries of $CD$ in (18a). Similarly for the second and third simple matrices.

We now show how an $m$-by-$n$ matrix $A$ can be decomposed into a sum of $k$ simple matrices, where $k = \text{rank}(A)$. Another way to say this is that we subtract a set of simple matrices from $A$ to eliminate all entries in $A$ (to reduce $A$ to the $O$ matrix).

Our strategy will be to form a simple matrix $K_1 = I_1 * u_1$ whose first row equals $a_1^R$ (the first row of $A$) and whose first column equals $a_1^C$ (the first column of $A$). Then $A - K_1$ will have $0$'s in its first row and column. We form $K_2$ to remove the second row and column of $A$; possibly we zero out additional rows and columns in the process. We continue similarly with $K_3$, and so on.

Let $u_1 = a_1^R$ and let
\[ l_i = \left( \frac{1}{a_{11}} \right) a_i^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & \cdots & \cdots \\ a_{31} & \cdots & \cdots \end{bmatrix} \]

(actually the first entry of \( l_i \) is \( 1, = a_{11}/a_{11} \)). Then the entries of first row of \( l_i \) * \( u_1 \) are 1 (first entry of \( l_i \)) times \( u_1 \), which equals \( u_i \) (= \( a_i^p \)), as required. And the entries in the first column of \( l_i \) * \( u_1 \) are \( l_i \) times the first entry of \( u_1 \), \( a_{11} \). We have

\[
\begin{bmatrix} 1 \\ \frac{a_{12}}{a_{11}} \\ \frac{a_{21}}{a_{11}} \\ \frac{a_{31}}{a_{11}} \end{bmatrix} \cdot [a_{11}, a_{12}, a_{13}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \cdots & \cdots \\ a_{31} & \cdots & \cdots \end{bmatrix}
\]

So \( A - K_1 \) \((= A - l_i \cdot u_i)\) has zeros in the first row and column.

Observe that outside the first row and column, the new entry \((i, j)\) in \( A - K_1 \) equals

\[ a_{ij} - \left( \frac{a_{1j}}{a_{11}} \right) a_{ij} \]

Surprise! This is our old friend the elimination operation [when we pivot on entry \((1, 1)\)]. Vector \( l_i \) is just the first column of the matrix \( L \) of elimination multipliers from the \( A = LU \) decomposition, and \( u_i \) is the first row of \( U \) (which equals the first row of \( A \)). So we have shown that when we subtruct from \( A \), the simple matrix \( l_i^T * u_i^T \) formed by \( l_i^T \) (the first column of \( L \)) and \( u_i^T \) (the first row of \( U \)), we obtain a matrix with 0's in the first row and first column. This new matrix is just the coefficient matrix (ignoring the first row of 0's) for the remaining \( n - 1 \) equations in \( Ax = b \) when we pivot on entry \((1, 1)\).

Repeating this argument, we let \( K_2 = l_2^T * u_2^T \) and subtracting \( K_2 \) from \( A - K_1 \) will have the effect of next pivoting on entry \((2, 2)\), and zeroing out the second row and column. The other \( K_i \) are defined and perform similarly. so ultimately we see that

\[ A = l_1^T * u_1^T + l_2^T * u_2^T + \cdots + l_n^T * u_n^T \]  \hspace{1cm} (19)

**Example 8. Refinery Matrix Expressed as a Sum of Simple Matrices**

In Section 3.2 we gave the \( LU \) decomposition of our refinery matrix

\[
A = \begin{bmatrix} 20 & 4 & 4 \\ 10 & 14 & 5 \\ 5 & 5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 20 & 4 & 4 \\ 0 & 12 & 3 \\ 0 & 0 & 10 \end{bmatrix}
\]
By (19), we can write \( \mathbf{A} \) as
\[
\mathbf{A} = \mathbf{k}_1 \mathbf{u}_1^\mathbf{f} + \mathbf{k}_2 \mathbf{u}_2^\mathbf{f} + \mathbf{k}_3 \mathbf{u}_3^\mathbf{f}
\]
\[
= \begin{bmatrix} 1 \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} 20 & 4 & 4 \\ 0 & 12 & 3 \\ 0 & 0 & 10 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
\]
\[
= \begin{bmatrix} 20 & 4 & 4 \\ 10 & 2 & 2 \\ 5 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 12 & 3 \\ 0 & 0 & 0 \end{bmatrix}
\]
\[
= \begin{bmatrix} 20 & 4 & 4 \\ 10 & 14 & 4 \\ 5 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 12 & 3 \\ 0 & 0 & 0 \end{bmatrix}
\]
\[
= \begin{bmatrix} 20 & 18 & 4 \\ 10 & 26 & 4 \\ 5 & 1 & 1 \end{bmatrix}
\]
(20)

The reader should check that this set of three simple matrices adds up to \( \mathbf{A} \).

**Theorem 8.** Gaussian elimination can be viewed as a decomposition of \( \mathbf{A} \) into a sum of rank(\( \mathbf{A} \)) simple matrices:
\[
\mathbf{A} = \mathbf{k}_1 \mathbf{u}_1^\mathbf{f} + \mathbf{k}_2 \mathbf{u}_2^\mathbf{f} + \cdots + \mathbf{k}_r \mathbf{u}_r^\mathbf{f}
\]
where \( k = \text{rank}(\mathbf{A}) \), \( \mathbf{k}_i \) is the \( i \)th column of \( \mathbf{L} \) (the matrix of elimination multipliers), and \( \mathbf{u}_i^\mathbf{f} \) is the \( i \)th row of \( \mathbf{U} \) (the reduced matrix in Gaussian elimination).

The minimum number of simple matrices whose sum equals matrix \( \mathbf{A} \) is rank(\( \mathbf{A} \)).

The last sentence of Theorem 8 is proved in Exercise 32. The symmetric role of columns and rows in Theorem 8 explains why the dimensions of the row and column spaces of a matrix are equal.

It is not hard to show (see Exercise 31) that if \( \mathbf{A} \) has the simple-matrix decomposition \( \mathbf{A} = \mathbf{c}_1 \mathbf{d}_1 + \cdots + \mathbf{c}_r \mathbf{d}_r \), then the \( \mathbf{c}_i \) are a basis for the column space of \( \mathbf{A} \) and the \( \mathbf{d}_i \) are a basis for the row space of \( \mathbf{A} \). It follows that

**Corollary**

(i) The nonzero rows of \( \mathbf{U} \) generate the row space of \( \mathbf{A} \) and the nonzero (below main diagonal) columns of \( \mathbf{L} \) generate the column space of \( \mathbf{A} \).

(ii) Theorem 8 reproves the fact that the dimension of the column space of \( \mathbf{A} \) equals the dimension of the row space of \( \mathbf{A} \), equals rank(\( \mathbf{A} \)).

From Theorem 7 it follows if \( \mathbf{A} \) equals the sum of simple matrices \( \mathbf{k}_i \mathbf{u}_i^\mathbf{f} \), then \( \mathbf{A} = \mathbf{LU} \)—we have proved the LU decomposition.

The decomposition (19) of a matrix \( \mathbf{A} \) into a sum of rank(\( \mathbf{A} \)) simple matrices is of more theoretical than practical interest.
Optional (Based on Section 4.6)

Let us reinterpret the simplex algorithm of linear programming using the concept of a basis for the column space. When slack variables were added, as say to the table–chair production problem in Section 4.6, the form of the constraint equations was

\[
\begin{bmatrix}
1 & 4 & 1 & 0 & 0 & 0 & \vdots & 1400 \\
2 & 3 & 0 & 1 & 0 & 0 & \vdots & 2000 \\
1 & 12 & 0 & 0 & 1 & 0 & \vdots & 3600 \\
2 & 0 & 0 & 0 & 0 & 1 & \vdots & 1800
\end{bmatrix}
\]

Observe that the columns associated with the slack variables \(x_3, x_4, x_5, x_6\) form an identity matrix and hence are the basis for the column space. For this reason, \(x_3, x_4, x_5, x_6\) are called basic variables and variables \(x_1, x_2\) nonbasic, for the linear program (22): Clearly, the columns of nonbasic variables \(x_1, x_2\) in (22) are linearly dependent on the basic variables’ columns. Recall that the simplex algorithm sets the nonbasic variables equal to 0 so that the basic variables then have nonnegative values equal to the corresponding right-side entry.

The pivot step in the simplex algorithm can be viewed as picking some nonbasic variable to enter the basis while a basic variable leaves the basis. For (22), we chose \(x_2\) (whose coefficient 200 in the objective function is largest) to enter and \(x_3\) to leave the basis. After pivoting, we have

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & -60,000 \\
\frac{5}{3} & 0 & 1 & 0 & \frac{-1}{3} & 0 & \vdots & 200 \\
0 & 0 & 0 & 1 & \frac{-1}{4} & 0 & \vdots & 1,100 \\
0 & 1 & 0 & 0 & \frac{1}{12} & 0 & \vdots & 300 \\
2 & 0 & 0 & 0 & 0 & 1 & \vdots & 1,800
\end{bmatrix}
\]

Note that now the columns of \(x_2, x_3, x_4, x_5\) form the basis for the column space.

Whereas our discussion in Section 4.6 focused on which were the independent (nonbasic) variables, the traditional approach is to concentrate on which are the basic variables.

Section 5.2 Exercises

Summary of Exercises
Exercises 1–10 involve the column space, linear dependence, and generators of the column space. Exercises 11–25 involve associated theory. Exercises 26–32 involve simple matrices and the representation of a matrix as a sum of simple matrices.
1. Three soup factories $F_1$, $F_2$, and $F_3$ generate production vectors, $\mathbf{f}_1 = [20, 100, 20]$, $\mathbf{f}_2 = [200, 0, 50]$, and $\mathbf{f}_3 = [0, 100, 200]$, of the amounts (in gallons) of tomato, chicken, and split-pea soup produced each hour. If the demand is $\mathbf{d} = [5000, 3000, 3000]$, write a system of equations for determining the right linear combination (how long each factory should work) of production vectors to meet the demand. Determine the weights.

2. There are three refineries producing heating oil, diesel oil, and gasoline. The production vector of refinery $A$ (per barrel of crude oil) is $[10, 5, 10]$ and of refinery $B$ is $[4, 11, 8]$. The production vector for refinery $C$ is the average of the vector for refineries $A$ and $B$. If the demand vector is $[380, 370, 460]$, write a system of equations for finding the right linear combination of refinery production vectors to equal the demand vector. Find the set of such linear combinations.

3. For each of the following sets of vectors, express the first vector as a linear combination of the remaining vectors if possible.
   (a) $[1, 1]; [2, 1]; [2, -1]$  
   (b) $[3, 2]; [2, -3]; [-3, 6]$
   (c) $[3, -1]; [1, 3]; [-2, 3]$
   (d) $[1, 1, 1]; [2, 1, 0]; [0, 1, 2]; [3, 2, 1]$

4. For each of the following pairs of a matrix and a vector, express the vector as a linear combination of the columns of the matrix. Plot this linear combination as was done in Figure 5.3.
   (a) $\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  
   (b) $\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}$  
   (c) $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

5. The first column in the inverse $A^{-1}$ of a 2-by-2 matrix $A$ gives the weights in a linear combination of $A$'s columns that equals $\mathbf{e}_1 = [1, 0]$. The second column in $A^{-1}$ gives the weights in a linear combination of $A$'s columns that equals $\mathbf{e}_2 = [0, 1]$. Find these weights for expressing $\mathbf{e}_1$ and $\mathbf{e}_2$ as linear combinations of the columns and plot the linear combinations as in Figure 5.3 for the following matrices.
   (a) $\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$  
   (b) $\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$  
   (c) $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$

6. Tell which of the following sets of vectors are linearly independent. If linearly dependent, express one vector as a linear combination of the others.
   (a) $[1, 2]; [-2, 4]$  
   (b) $[1, 3]; [3, -1]$  
   (c) $[2, 1]; [2, 3]; [2, 8]$  
   (d) $[1, 1, 1]; [-2, 0, -2]; [2, 1, 2]$  
   (e) $[2, 1, 0]; [1, 1, 3]; [0, 2, 1]$

7. Find a set of columns that form a basis for the column space of each of the following matrices (use the reduced matrix $A*$ as in Examples 3 and 4). Give the rank of each matrix.
8. For each matrix in Exercise 7, find all sets of columns that form a basis for the column space.

9. Find a set of columns that form a basis for the column space of each of the following matrices. Give the rank of each matrix. Also find a basis for the null space of each matrix.

(a) $\begin{bmatrix} -3 & 5 \\ 6 & -10 \end{bmatrix}$
(b) $\begin{bmatrix} 2 & 1 & 7 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$
(d) $\begin{bmatrix} 1 & 0 & 2 & -1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & -1 & 2 & 1 \\ 1 & 0 & 2 & -1 & 1 \end{bmatrix}$

10. Let $A$ be a coefficient matrix in a refinery problem, as in Example 1, with each column representing the production vector of a refinery. Explain the practical significance of having one column be a linear combination of the others. What constraints and what freedom does this permit the manager of the refineries?

11. For a $m$-by-$n$ matrix $A$, the reduced matrix $A^\ast$ can be written in the partitioned form $A^\ast = \begin{bmatrix} I & R \\ O & O \end{bmatrix}$, where $I$ is an $r$-by-$r$ identity matrix ($r = \text{rank}(A)$) and $R$ is $r$-by-$(n - r)$. Using the submatrix $R$ and an appropriate size identity matrix $I$, give a matrix $N$ in partitioned form whose columns are the basis of $\text{Null}(A)$.

*Hint:* See expression (14).
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12. Determine the rank of matrix \( A \), if possible, from the given information.
   (a) \( A \) is an \( n \)-by-\( n \) matrix with linearly independent columns.
   (b) \( A \) is a 6-by-4 matrix and \( \text{Null}(A) = \{0\} \).
   (c) \( A \) is a 5-by-6 matrix and \( \dim(\text{Null}(A)) = 3 \).
   (d) \( A \) is a 3-by-3 matrix and \( \det(A) = 17 \).
   (e) \( A \) is a 5-by-5 and \( \dim(\text{Row}(A)) = 3 \).
   (f) \( A \) is an invertible 4-by-4 matrix.
   (g) \( A \) is a 4-by-3 matrix and \( Ax = b \) has either a unique solution or else no solution.
   (h) \( A \) is a 8-by-8 matrix and \( \dim(\text{Row}(A^T)) = 6 \).
   (i) \( A \) is a 7-by-5 matrix in which \( \dim(\text{Null}(A^T)) = 3 \).

13. In this exercise, the reader should try to find by inspection a linear dependence among the rows of each matrix. If dependence is found, use elimination by pivoting to find a linear dependence among the columns (as in Examples 3 and 4).

   \[
   \begin{bmatrix}
   5 & 2 & 3 \\
   0 & 1 & 2 \\
   5 & 3 & 5 \\
   \end{bmatrix} \quad \begin{bmatrix}
   1 & 2 & 0 \\
   1 & 1 & 1 \\
   3 & 4 & 2 \\
   \end{bmatrix} \quad \begin{bmatrix}
   5 & 7 & 9 \\
   4 & 5 & 6 \\
   1 & 2 & 3 \\
   \end{bmatrix}
   \]

14. Show that the number of pivots performed in Gaussian elimination will be the same as the number of pivots in elimination by (full) pivoting. (Thus the rank of a matrix can be defined in terms of either type of elimination.)

15. Let \( A^* \) be the reduced-form matrix of \( A \).
   (a) Show that nonzero rows of \( A^* \) generate \( \text{Row}(A) \) (i.e., linear combinations of the rows of \( A^* \) generate the same vectors as linear combinations of rows of \( A \)).
   (b) Show that the nonzero rows of \( A^* \) must be linearly independent.
   \( \text{Hint:} \) Look at the form of \( A^* \).
   (c) Conclude that the nonzero rows of \( A^* \) are a basis of \( \text{Row}(A) \) and hence that \( \dim(\text{Row}(A)) = \text{rank}(A) \).

16. Let \( U \) be the upper triangular matrix produced at the end of Gaussian elimination on the matrix \( A \).
   (a) Show that nonzero rows of \( U \) generate \( \text{Row}(A) \) (i.e., linear combinations of the rows of \( U \) generate the same vectors as linear combinations of rows of \( A \)).
   (b) Show that the nonzero rows of \( U \) must be linearly independent.
   \( \text{Hint:} \) Look at the form of \( U \).
   (c) Conclude that the nonzero rows of \( U \) are a basis of \( \text{Row}(A) \) and hence that \( \dim(\text{Row}(A)) = \text{rank}(A) \).

17. (a) Suppose that the rows of \( A \) are linearly dependent. Show that at the end of Gaussian elimination, the resulting upper triangular matrix \( U \) will have at least one row of zeros.
   (b) Suppose that \( A \) is a square matrix with linearly dependent columns.
Show that at the end of Gaussian elimination, the resulting matrix $U$ will have at least one row of zeros.

**Hint:** Use part (a) and Theorem 5.

18. Show that $\text{Row}(A^T)$ ($A^T$ is the transpose of $A$) equals the $\text{Col}(A)$ and that $\text{Col}(A^T)$ equals $\text{Row}(A)$. Show that $\text{rank}(A) = \text{rank}(A^T)$.

19. (a) Use the results of Exercise 17 and 18 to show that the nonzero rows in the reduced form $A^* = A^T$ are a basis for the $\text{Col}(A)$.
   (b) Use part (a) to compute a basis for $\text{Col}(A)$ for the matrix $A$ in Example 3.
   (c) Repeat part (b) for the matrix in Example 4.

20. (a) Show that the rank of a matrix does not change when a multiple of one row is subtracted from another row.
   (b) Show that the rank of a matrix does not change when a multiple of one column is subtracted from another column.

**Hint:** Use $A^T$.

21. (a) Show that if $A$ is an $m$-by-$n$ matrix and $b$ an $m$-vector, $b$ is in $\text{Range}(A) \{= \text{Col}(A)\}$ if and only if $\text{rank}([A \ b]) = \text{rank}(A)$, where $[A \ b]$ denotes the augmented $m$-by-$(n+1)$ matrix with $b$ added as an extra column to $A$.
   (b) If $Ax = b$ has no solution, show that $\text{rank}([A \ b])$ must be $\text{rank}(A) + 1$.

22. Let $A$ be an $n$-by-$n$ matrix. Show that $\text{det}(A) = 0$ if and only if the rows of $A$ are linearly dependent or if the columns of $A$ are linearly dependent.

**Hint:** Use Theorem 5 and Theorem 4 of Section 3.3.

23. This exercise examines the vectors in the column space of two matrices $A$ and $B$, that is, vectors in $\text{Col}(A) \cap \text{Col}(B)$. If $d$ is such a vector, then $Ax = d$ and $Bx = d$, for some $x'$, $x''$. Show that if $C = [A \ -B]$ and $x^* = [x' \ x'']$, then $d$ is in $\text{Col}(A) \cap \text{Col}(B)$ if and only if $x^*$ is in $\text{Null}(C)$.

24. Show that any set $H$ of $k$ linearly independent $n$-vectors, $k < n$, can be extended to a basis for all $n$-vectors.

**Hint:** Form an $n$-by-$(k + n)$ matrix $A$ whose first $k$ columns come from $H$ and whose last $n$ columns are the identity matrix—thus $\dim(\text{Col}(A)) = n$; show that a basis for $\text{Col}(A)$ using the elimination by pivoting approach in Example 5 will include the columns of $H$.

25. Show that $\lambda$ is an eigenvalue of $A$ if and only if $\text{det}(A - \lambda I) = 0$.

**Hint:** “If” part is immediate; for the “only if” part, use Theorem 3 and Theorem 4 of Section 3.3.
26. Let \( \mathbf{a} = [1, 2, 3], \mathbf{b} = [2, 0], \mathbf{c} = [-1, 2, 1] \). Compute the following simple matrices.
   (a) \( \mathbf{a} \cdot \mathbf{b} \)  
   (b) \( \mathbf{a} \cdot \mathbf{c} \)  
   (c) \( \mathbf{c} \cdot \mathbf{c} \)

27. Verify that the simple matrices in Exercise 26 have rank 1.

28. (a) Show that \( \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \) is a simple matrix by giving the two vectors whose matrix product is \( \mathbf{A} \).
   (b) Repeat part (a) for \( \mathbf{B} = \begin{bmatrix} 12 & -6 & 9 \\ 8 & -4 & 6 \\ 4 & -2 & 3 \end{bmatrix} \)

29. Write each of the following matrices as the sum of two simple matrices.
   (a) \( \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \)  
   (b) \( \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 7 & 8 & 9 \end{bmatrix} \)

30. (a) Find the LU decomposition of matrix in Exercise 9, part (b) and use the decomposition to write the matrix as the sum of two simple matrices as in Example 8.
   (b) Repeat part (a) for the matrix in Example 5.
   (c) Repeat part (a) for the matrix in Exercise 9, part (d).

31. Describe the column space and row space of a simple matrix \( \mathbf{a} \cdot \mathbf{b} \) and give a basis for each.

32. (a) Show that if a matrix \( \mathbf{A} \) of rank \( k \) is expressed as the sum of \( k \) simple matrices \( \mathbf{c}_i \cdot \mathbf{d}_i \), \( i = 1, 2, \ldots, k \), then the \( \mathbf{c}_i \) are a basis of \( \text{Col}(\mathbf{A}) \) and the \( \mathbf{d}_i \) are a basis of \( \text{Row}(\mathbf{A}) \).
   (b) Prove the last sentence in Theorem 8, the minimum number of simple matrices whose sum is matrix \( \mathbf{A} \) is \( \text{rank}(\mathbf{A}) \), as follows: The \( \text{LU} \) decomposition yields a sum of \( k \) simple matrices equaling \( \mathbf{A} \), where \( k = \text{rank}(\mathbf{A}) \), by the first part of Theorem 8; if fewer than \( k \) simple matrices could sum to \( \mathbf{A} \), use part (a) to show that then \( \dim(\text{Col}(\mathbf{A})) < \text{rank}(\mathbf{A}) \) —impossible.