Chapter 2

Definitions and examples

I hate definitions!
Benjamin Disraeli

In this chapter, we lay the foundations for a proper study of graph theory. Section 2 formalizes some of the basic definitions of Chapter 1 and Section 3 provides a variety of examples. In Section 4 we show how graphs can be used to represent and solve three problems from recreational mathematics. More substantial applications are deferred until we have more machinery at our disposal (see Sections 8 and 11).

2 Definitions

A **simple graph** G consists of a non-empty finite set $V(G)$ of elements called **vertices** (or **nodes**), and a finite set $E(G)$ of distinct unordered pairs of distinct elements of $V(G)$ called **edges**. We call $V(G)$ the **vertex set** and $E(G)$ the **edge set** of G. An edge $\{v, w\}$ is said to **join** the vertices v and w, and is usually abbreviated to vw. For example, Fig. 2.1 represents the simple graph G whose vertex set $V(G)$ is $\{u, v, w, z\}$, and whose edge set $E(G)$ consists of the edges uv, uw, vw and wz.

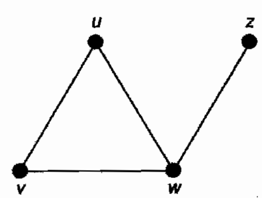

Fig. 2.1

In any simple graph there is at most one edge joining a given pair of vertices. However, many results that hold for simple graphs can be extended to more general objects in which two vertices may have several edges joining them. In addition, we may remove the restriction that an edge joins two *distinct* vertices, and allow **loops** – edges joining a vertex to itself. The resulting object, in which loops and multiple edges are allowed, is called a **general graph** – or, simply, a **graph** (see Fig. 2.2). Thus every simple graph is a graph, but not every graph is a simple graph.

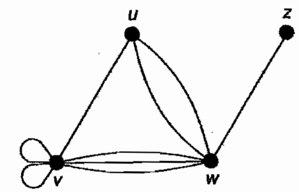

Fig. 2.2

Thus, a **graph** G consists of a non-empty finite set $V(G)$ of elements called **vertices**, and a finite family $E(G)$ of unordered pairs of (not necessarily distinct) elements of $V(G)$ called **edges**; the use of the word 'family' permits the existence of multiple edges[†]. We call $V(G)$ the **vertex set** and $E(G)$ the **edge family** of G. An edge $\{v, w\}$ is said to **join** the vertices v and w, and is again abbreviated to vw. Thus in Fig. 2.2, $V(G)$ is the set $\{u, v, w, z\}$ and $E(G)$ consists of the edges uv, vv (twice), vw (three times), uw (twice), and wz. Note that each loop vv joins the vertex v to itself. Although we sometimes have to restrict our attention to simple graphs, we shall prove our results for general graphs whenever possible.

The language of graph theory is not standard – all authors have their own terminology. Some use the term 'graph' for what we call a simple graph, or for a graph with directed edges, or for a graph with infinitely many vertices or edges; we discuss digraphs in Chapter 7 and infinite graphs in Section 16. Any such definition is perfectly valid, provided that it is used consistently. In this book, *all graphs are finite and undirected, with loops and multiple edges allowed unless specifically excluded.*

Isomorphism

Two graphs G_1 and G_2 are **isomorphic** if there is a one–one correspondence between the vertices of G_1 and those of G_2 such that the number of edges joining any two vertices of G_1 is equal to the number of edges joining the corresponding vertices of G_2. Thus the two graphs shown in Fig. 2.3 are isomorphic under the correspondence $u \leftrightarrow l$, $v \leftrightarrow m$, $w \leftrightarrow n$, $x \leftrightarrow p$, $y \leftrightarrow q$, $z \leftrightarrow r$. For many problems, the labels on the vertices are unnecessary and we drop them. We then say that two 'unlabelled graphs' are isomorphic if we can assign labels so that the resulting 'labelled graphs' are

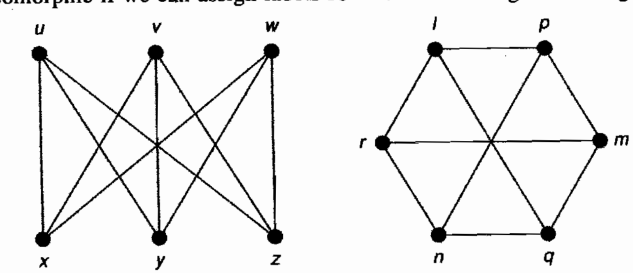

Fig. 2.3

† We use the word 'family' to mean a collection of elements, some of which may occur several times; for example, $\{a, b, c\}$ is a set, but (a, a, c, b, a, c) is a family.

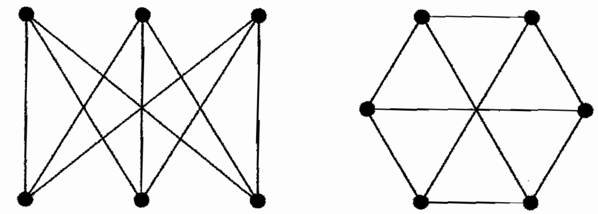

Fig. 2.4

isomorphic. For example, the unlabelled graphs in Fig. 2.4 are isomorphic, since we can label the vertices as in Fig. 2.3.

The difference between labelled and unlabelled graphs becomes more apparent when we try to count them. For example, if we restrict ourselves to graphs with three vertices, then there are, up to isomorphism, eight different labelled graphs but only four unlabelled ones (see Figs 2.5 and 2.6). It is usually clear from the context whether we are referring to labelled or unlabelled graphs.

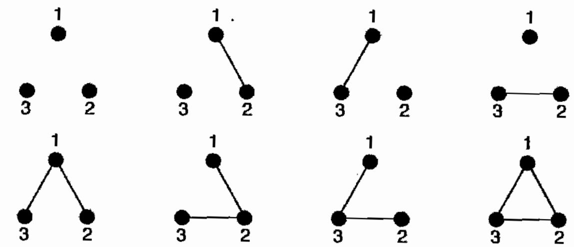

Fig. 2.5

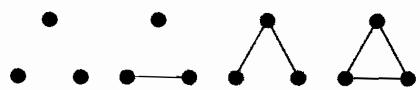

Fig. 2.6

Connectedness

We can combine two graphs to make a larger graph. If the two graphs are $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, where $V(G_1)$ and $V(G_2)$ are disjoint, then their **union** $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge family $E(G_1) \cup E(G_2)$ (see Fig. 2.7).

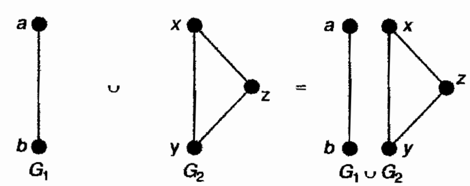

Fig. 2.7

Most all the graphs discussed so far have been 'in one piece'. A graph is **connected** if it cannot be expressed as the union of two graphs, and **disconnected** otherwise. Clearly any disconnected graph G can be expressed as the union of connected graphs,

Adjacency

We say that two vertices v and w of a graph G are **adjacent** if there is an edge vw joining them, and the vertices v and w are then **incident** with such an edge. Similarly, two distinct edges e and f are **adjacent** if they have a vertex in common (see Fig. 2.10).

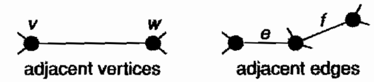

Fig. 2.10

The **degree** of a vertex v of G is the number of edges incident with v, and is written deg(v); in calculating the degree of v, we usually make the convention that a loop at v contributes 2 (rather than 1) to the degree of v. A vertex of degree 0 is an **isolated vertex** and a vertex of degree 1 is an **end-vertex**. Thus each of the two graphs in Fig. 2.11 has two end-vertices and three vertices of degree 2, while the graph in Fig. 2.12 has one end-vertex, one vertex of degree 3, one of degree 6 and one of degree 8. The **degree sequence** of a graph consists of the degrees written in increasing order, with repeats where necessary. For example, the degree sequences of the graphs in Figs. 2.11 and 2.12 are (1, 1, 2, 2, 2) and (1, 3, 6, 8).

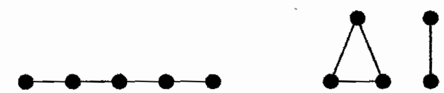

Fig. 2.11

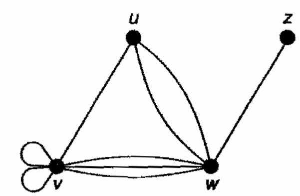

Fig. 2.12

Note that *in any graph the sum of all the vertex-degrees is an even number* – in fact, twice the number of edges, since each edge contributes exactly 2 to the sum. This result, due essentially to Leonhard Euler in 1736, is called the **handshaking lemma**. It implies that if several people shake hands, then the total number of hands shaken must be even – precisely because just two hands are involved in each handshake. An immediate corollary of the handshaking lemma is that *in any graph the number of vertices of odd degree is even.*

Subgraphs

A **subgraph** of a graph G is a graph, each of whose vertices belongs to $V(G)$ and each of whose edges belongs to $E(G)$. Thus the graph in Fig. 2.13 is a subgraph of the graph in Fig. 2.14, but is not a subgraph of the graph in Fig. 2.15, since the latter graph contains no 'triangle'.

2.3[s] (i) By suitably labelling the vertices, show that the two graphs in Fig. 2.20 are isomorphic.
(ii) Explain why the two graphs in Fig. 2.21 are not isomorphic.

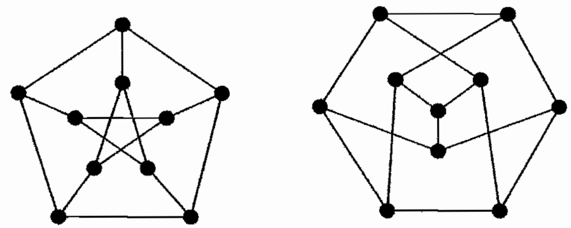

Fig. 2.20

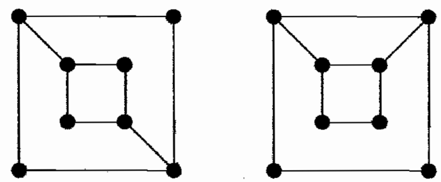

Fig. 2.21

2.4 Classify the following statements as *true* or *false:*
(i) any two isomorphic graphs have the same degree sequence;
(ii) any two graphs with the same degree sequence are isomorphic.

2.5 (i) Show that there are exactly $2^{n(n-1)/2}$ labelled simple graphs on n vertices.
(ii) How many of these have exactly m edges?

2.6[s] Locate each of the graphs in Fig. 2.22 in the table of Fig. 2.9.

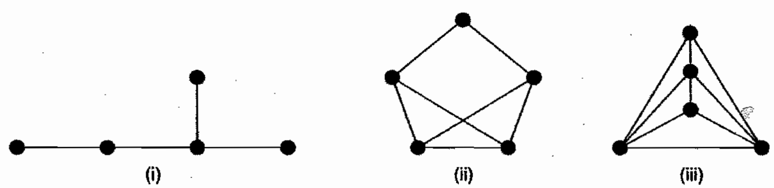

Fig. 2.22

2.7[s] Write down the degree sequence of each graph with four vertices in Fig. 2.9, and verify that the handshaking lemma holds for each graph.

2.8 (i) Draw a graph on six vertices with degree sequence (3, 3, 5, 5, 5, 5); does there exist a *simple* graph with these degrees?
(ii) How are your answers to part (i) changed if the degree sequence is (2, 3, 3, 4, 5, 5)?

2.9* If G is a simple graph with at least two vertices, prove that G must contain two or more vertices of the same degree.

2.10[s] Which graphs in Fig. 2.23 are subgraphs of those in Fig. 2.20?

3 Examples

In this section we examine some important types of graphs. You should become familiar with them, as they will appear frequently in examples and exercises.

Null graphs

A graph whose edge-set is empty is a **null graph**. We denote the null graph on n vertices by N_n; N_4 is shown in Fig. 3.1. Note that each vertex of a null graph is isolated. Null graphs are not very interesting.

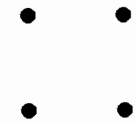

Fig. 3.1

Complete graphs

A simple graph in which each pair of distinct vertices are adjacent is a **complete graph**. We denote the complete graph on n vertices by K_n; K_4 and K_5 are shown in Fig. 3.2. You should check that K_n has $n(n-1)/2$ edges.

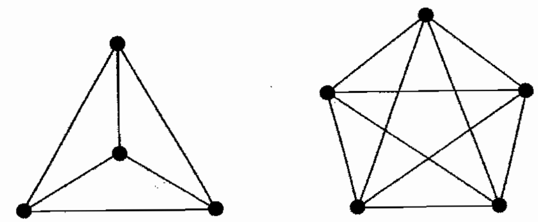

Fig. 3.2

Cycle graphs, path graphs and wheels

A connected graph that is regular of degree 2 is a **cycle graph**. We denote the cycle graph on n vertices by C_n. The graph obtained from C_n by removing an edge is the **path graph** on n vertices, denoted by P_n. The graph obtained from C_{n-1} by joining each vertex to a new vertex v is the **wheel** on n vertices, denoted by W_n. The graphs C_6, P_6 and W_6 are shown in Fig. 3.3.

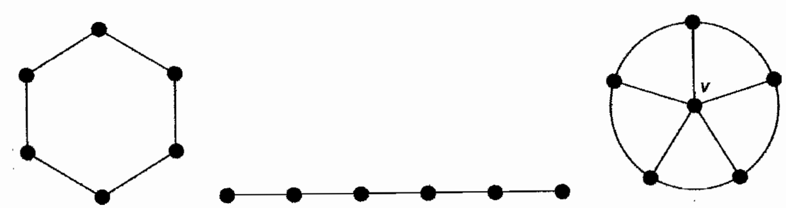

Fig. 3.3

The complement of a simple graph

If G is a simple graph with vertex set $V(G)$, its **complement** $\bar{G}$ is the simple graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are *not* adjacent in G. For example, Fig. 3.9 shows a graph and its complement. Note that the complement of a complete graph is a null graph, and that the complement of a complete bipartite graph is the union of two complete graphs.

Exercises 3

3.1[s] Draw the following graphs:
- (i) the null graph N_5;
- (ii) the complete graph K_6;
- (iii) the complete bipartite graph $K_{2,4}$;
- (iv) the union of $K_{1,3}$ and W_4;
- (v) the complement of the cycle graph C_4.

3.2[s] How many edges has each of the following graphs:
(i) K_{10}; (ii) $K_{5,7}$; (iii) Q_4; (iv) W_8; (v) the Petersen graph?

3.3 How many vertices and edges has each of the Platonic graphs?

3.4[s] In the table of Fig. 2.9, locate all the regular graphs and the bipartite graphs.

3.5 Give an example (if it exists) of each of the following:
- (i) a bipartite graph that is regular of degree 5;
- (ii) a bipartite Platonic graph;
- (iii) a complete graph that is a wheel;
- (iv) a cubic graph with 11 vertices;
- (v) a graph (other than K_4, $K_{4,4}$ or Q_4) that is regular of degree 4.

3.6[s] Draw all the simple cubic graphs with at most 8 vertices.

3.7 The **complete tripartite graph** $K_{r,s,t}$ consists of three sets of vertices (of sizes r, s and t), with an edge joining two vertices if and only if they lie in different sets. Draw the graphs $K_{2,2,2}$ and $K_{3,3,2}$ and find the number of edges of $K_{3,4,5}$.

3.8 A simple graph that is isomorphic to its complement is **self-complementary**.
- (i) Prove that, if G is self-complementary, then G has $4k$ or $4k+1$ vertices, where k is an integer.
- (ii) Find all self-complementary graphs with 4 and 5 vertices.
- (iii) Find a self-complementary graph with 8 vertices.

3.9* The **line graph** $L(G)$ of a simple graph G is the graph whose vertices are in one–one correspondence with the *edges* of G, two vertices of $L(G)$ being adjacent if and only if the corresponding edges of G are adjacent.
- (i) Show that K_3 and $K_{1,3}$ have the same line graph.
- (ii) Show that the line graph of the tetrahedron graph is the octahedron graph.
- (iii) Prove that, if G is regular of degree k, then $L(G)$ is regular of degree $2k-2$.
- (iv) Find an expression for the number of edges of $L(G)$ in terms of the degrees of the vertices of G.
- (v) Show that $L(K_5)$ is the complement of the Petersen graph.

3.10* An **automorphism** φ of a simple graph G is a one–one mapping of the vertex set of G onto itself with the property that $\varphi(v)$ and $\varphi(w)$ are adjacent whenever v and w are. The **automorphism group** $\Gamma(G)$ of G is the group of automorphisms of G under composition.

Regular graphs

A graph in which each vertex has the same degree is a **regular graph**. If each vertex has degree r, the graph is **regular of degree** r or r**-regular**. Of special importance are the **cubic graphs**, which are regular of degree 3; an example of a cubic graph is the **Petersen graph**, shown in Fig. 3.4. Note that the null graph N_n is regular of degree 0, the cycle graph C_n is regular of degree 2, and the complete graph K_n is regular of degree $n-1$.

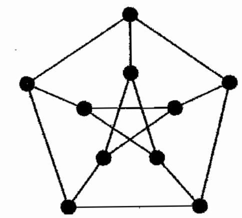

Fig. 3.4

Platonic graphs

Of interest among the regular graphs are the **Platonic graphs**, formed from the vertices and edges of the five regular (Platonic) solids – the tetrahedron, octahedron, cube, icosahedron and dodecahedron (see Fig. 3.5).

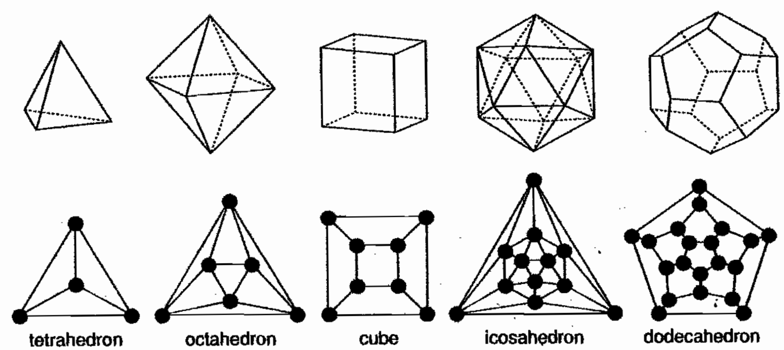

Fig. 3.5

Bipartite graphs

If the vertex set of a graph G can be split into two disjoint sets A and B so that each edge of G joins a vertex of A and a vertex of B, then G is a **bipartite graph** (see Fig. 3.6). Alternatively, a bipartite graph is one whose vertices can be coloured black and white in such a way that each edge joins a black vertex (in A) and a white vertex (in B).

A **complete bipartite graph** is a bipartite graph in which each vertex in A is joined to each vertex in B by just one edge. We denote the bipartite graph with r black vertices and s white vertices by $K_{r,s}$; $K_{1,3}$, $K_{2,3}$, $K_{3,3}$ and $K_{4,3}$ are shown in Fig. 3.7. You should check that $K_{r,s}$ has $r+s$ vertices and rs edges.

Chapter 5

Planarity

Flattery will get you nowhere.
Popular saying

We now embark upon a study of topological graph theory, in which graphs become tied up with topological notions such as planarity, genus, etc. In particular, we investigate when a graph can be drawn in the plane and on other surfaces. In Section 12, we discuss planar graphs, prove that some graphs are not planar, and state Kuratowski's characterization of planar graphs. In Section 13, we prove Euler's formula relating the numbers of vertices, edges and faces of a graph drawn in the plane, and generalize it to graphs drawn on other surfaces in Section 14. In Section 15 we study duality, and the chapter concludes, in Section 16, with some material on infinite graphs.

12 Planar graphs

A **planar graph** is a graph that can be drawn in the plane without crossings – that is, so that no two edges intersect geometrically except at a vertex to which both are incident. Any such drawing is a **plane drawing.** For convenience, we often use the abbreviation **plane graph** for a plane drawing of a planar graph. For example, Fig. 12.1 shows three drawings of the planar graph K_4, but only the second and third are plane graphs.

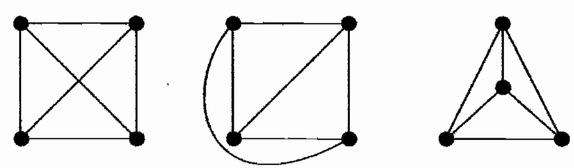

Fig. 12.1

The right-hand drawing in Fig. 12.1 leads us to ask whether every planar graph can be drawn in the plane so that each edge is represented by a straight line. Although loops or multiple edges cannot be drawn as straight lines, it was proved independently by K. Wagner in 1936 and I. Fáry in 1948 that *every simple planar graph can be drawn with straight lines*; see Chartrand and Lesniak [8] for details.

Not all graphs are planar, as the following theorem shows:

THEOREM 12.1. *$K_{3,3}$ and K_5 are non-planar.*

Remark. We give two proofs of this result. The first one, presented here, is constructive. The second proof, which we give in Section 13, appears as a corollary of Euler's formula.

Proof. Suppose first that $K_{3,3}$ is planar. Since $K_{3,3}$ has a cycle $u \to v \to w \to x \to y \to z \to u$ of length 6, any plane drawing must contain this cycle drawn in the form of a hexagon, as in Fig. 12.2.

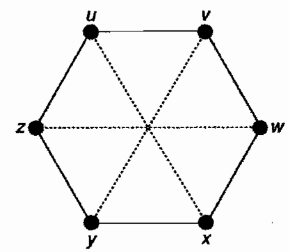

Fig. 12.2

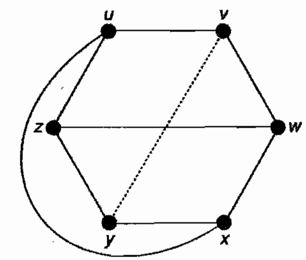

Fig. 12.3

Now the edge wz must lie either wholly inside the hexagon or wholly outside it. We deal with the case in which wz lies inside the hexagon – the other case is similar. Since the edge ux must not cross the edge wz, it must lie outside the hexagon; the situation is now as in Fig. 12.3. It is then impossible to draw the edge vy, as it would cross either ux or wz. This gives the required contradiction.

Now suppose that K_5 is planar. Since K_5 has a cycle $v \to w \to x \to y \to z \to v$ of length 5, any plane drawing must contain this cycle drawn in the form of a pentagon, as in Fig. 12.4.

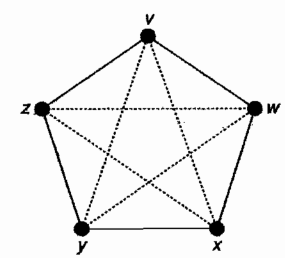

Fig. 12.4

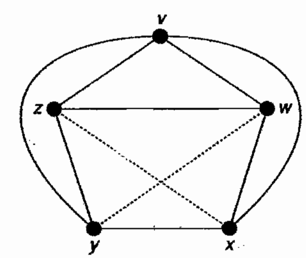

Fig. 12.5

Now the edge wz must lie either wholly inside the pentagon or wholly outside it. We deal with the case in which wz lies inside the pentagon – the other case is similar. Since the edges vx and vy do not cross the edge wz, they must both lie outside the pentagon; the situation is now as in Fig. 12.5. But the edge xz cannot cross the edge vy and so must lie inside the pentagon; similarly the edge wy must lie inside the pentagon, and the edges wy and xz must then cross. This gives the required contradiction. //

Note that every subgraph of a planar graph is planar, and that every graph with a non-planar subgraph must be non-planar. It follows that any graph with $K_{3,3}$ or K_5 as a subgraph is non-planar. In fact, as we shall see, these two graphs are the 'building blocks' for non-planar graphs, in the sense that every non-planar graph must 'contain' at least one of them.

To make this statement more precise, we define two graphs to be **homeomorphic** if both can be obtained from the same graph by inserting new vertices of degree 2 into its edges. For example, any two cycle graphs are homeomorphic, as are the graphs of Fig. 12.6.

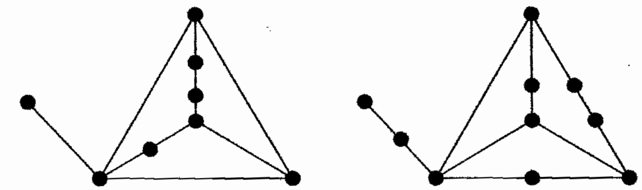

Fig. 12.6

Note that the introduction of the term 'homeomorphic' is merely a technicality, as the insertion or deletion of vertices of degree 2 is irrelevant to considerations of planarity. However, it enables us to state the following important result, known as **Kuratowski's theorem,** which gives a necessary and sufficient condition for a graph to be planar.

THEOREM 12.2 (Kuratowski, 1930). *A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.*

The proof of Kuratowski's theorem is long and involved, and we omit it; see Bondy and Murty [7] or Harary [9] for a proof. We shall, however, use Kuratowski's theorem to obtain another criterion for planarity. To do so, we first define a graph H to be **contractible** to K_5 or $K_{3,3}$ if we can obtain K_5 or $K_{3,3}$ by successively contracting edges of H. For example, the Petersen graph is contractible to K_5, as we can see by contracting the five 'spokes' joining the inner and outer 5-cycles (see Fig. 12.7).

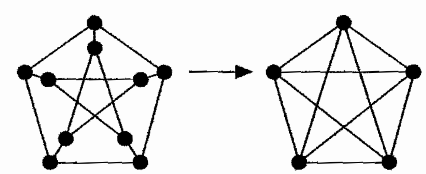

Fig. 12.7

THEOREM 12.3. *A graph is planar if and only if it contains no subgraph contractible to K_5 or $K_{3,3}$.*

Sketch of proof. $\Leftarrow$ Assume first that the graph G is non-planar. Then, by Kuratowski's theorem, G contains a subgraph H homeomorphic to K_5 or $K_{3,3}$. On successively contracting edges of H that are incident to a vertex of degree 2, we see that H is contractible to K_5 or $K_{3,3}$.

$\Rightarrow$ Now assume that G contains a subgraph H contractible to $K_{3,3}$, and let the vertex v of $K_{3,3}$ arise from contracting the subgraph H_v of H (see Fig. 12.8).

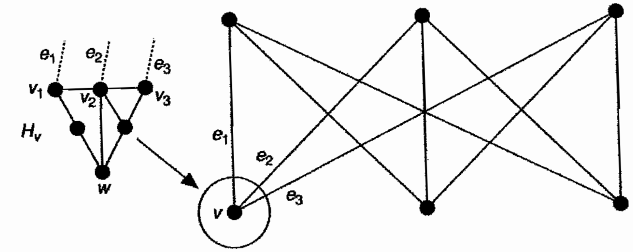

Fig. 12.8

The vertex v is incident in $K_{3,3}$ to three edges e_1, e_2 and e_3. When regarded as edges of H, these edges are incident to three (not necessarily distinct) vertices v_1, v_2 and v_3 of H_v. If v_1, v_2 and v_3 are distinct, then we can find a vertex w of H_v and three paths from w to these vertices, intersecting only at w. (There is a similar construction if the vertices are not distinct, the paths degenerating in this case to single vertices.) It follows that we can replace the subgraph H_v by a vertex w and three paths leading out of it. If this construction is carried out for each vertex of $K_{3,3}$, and the resulting paths joined up with the corresponding edges of $K_{3,3}$, then the resulting subgraph is homeomorphic to $K_{3,3}$. It follows from Kuratowski's theorem that G is non-planar (see Fig. 12.9).

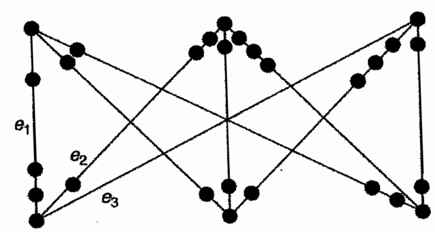

Fig. 12.9

A similar argument can be carried out if G contains a subgraph contractible to K_5. Here the details are more complicated, as the subgraph we obtain by the above process can be homeomorphic to either K_5 or $K_{3,3}$; see Chartrand and Lesniak [8]. //

We conclude this section by introducing the 'crossing-number' of a graph. If we try to draw K_5 or $K_{3,3}$ on the plane, then there must be at least one crossing of edges, since these graphs are not planar. However, we do not need more than one crossing (see Fig. 12.10), and we say that K_5 and $K_{3,3}$ have crossing number 1.

More generally, the **crossing number** cr(G) of a graph G is the minimum number of crossings that can occur when G is drawn in the plane. Thus, the crossing number

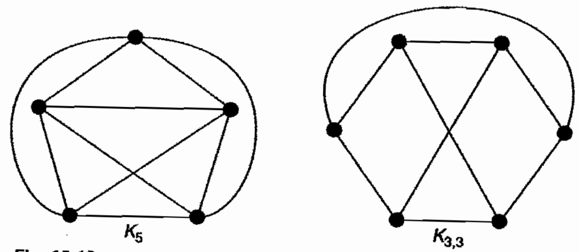

Fig. 12.10

can be used to measure how 'unplanar' G is. For example, the crossing number of any planar graph is 0, and $cr(K_5) = cr(K_{3,3}) = 1$. Note that the word 'crossing' always refers to the intersection of just two edges; crossings of three or more edges are not permitted.

Exercises 12

12.1^s Show, by drawing, that the following graphs are planar:
(i) the wheel W_5;
(ii) the graph of the octahedron.

12.2 Show how the graph of Fig. 12.11 can be drawn in the plane without crossings.

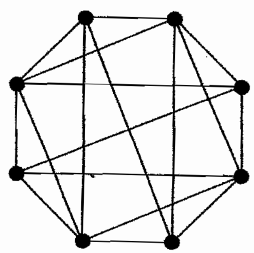

Fig. 12.11

12.3^s Three unfriendly neighbours use the same water, oil and treacle wells. In order to avoid meeting, they wish to build non-crossing paths from each of their houses to each of the three wells. Can this be done?

12.4^s Which complete graphs and complete bipartite graphs are planar?

12.5 (i) For which values of k is the k-cube Q_k planar?
(ii) For which values of r, s and t is the complete tripartite graph $K_{r,s,t}$ planar?

12.6 Prove that the Petersen graph is non-planar
(i) by using Theorem 12.2;
(ii) by using Theorem 12.3.
(Hint for part (i): remove the two 'horizontal' edges.)

12.7^s Give an example of
(i) a non-planar graph that is not homeomorphic to K_5 or $K_{3,3}$;
(ii) a non-planar graph that is not contractible to K_5 or $K_{3,3}$.
Why does the existence of these graphs not contradict Theorems 12.2 and 12.3?

12.8 Given that two homeomorphic graphs have n_i vertices and m_i edges ($i = 1, 2$), show that $m_1 - n_1 = m_2 - n_2$.

12.9 A graph G is **outerplanar** if G can be drawn in the plane so that all of its vertices lie on the exterior boundary.
(i) Show that K_4 and $K_{2,3}$ are not outerplanar.
(ii) Deduce that, if G is an outerplanar graph, then G contains no subgraph homeomorphic or contractible to K_4 or $K_{2,3}$. (The converse result also holds, yielding a Kuratowski-type criterion for a graph to be outerplanar.)

12.10^s Show that $K_{4,3}$ and the Petersen graph each has crossing number 2.

12.11* Given that r and s are both even, prove that

$$cr(K_{r,s}) \le rs(r-2)(s-2)/16,$$

and obtain corresponding results when r and/or s is odd.
(Hint: place the r vertices along the x-axis with $r/2$ vertices on each side of the origin, place the s vertices along the y-axis in a similar way, and count the crossings.)

12.12* Let G be a planar graph with vertex set $\{v_1, \ldots, v_n\}$, and let $p_1, \ldots, p_n$ be any n distinct points in the plane. Give a heuristic argument to show that G can be drawn in the plane in such a way that the point p_i represents the vertex v_i, for each i.

12.13* By placing the vertices at the points $(1, 1^2, 1^3), (2, 2^2, 2^3), (3, 3^2, 3^3), \ldots$, prove that any simple graph can be drawn without crossings in Euclidean three-dimensional space so that each edge is represented by a straight line.

13 Euler's formula

If G is a planar graph, then any plane drawing of G divides the set of points of the plane not lying on G into regions, called **faces**. For example, the plane graphs in Figs 13.1 and 13.2 have eight faces and four faces, respectively. Note that, in each case, the face f_4 is unbounded; it is called the **infinite face**.

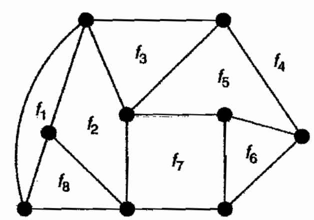

Fig. 13.1

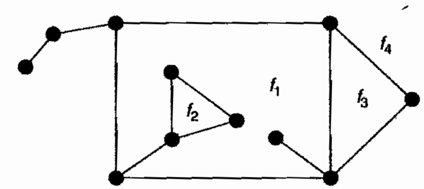

Fig. 13.2

There is nothing special about the infinite face – in fact, any face can be chosen as the infinite face. To see this, we map the graph onto the surface of a sphere by stereographic projection (see Fig. 13.3). We now rotate the sphere so that the point of projection (the north pole) lies inside the face we want as the infinite face, and then project the graph down onto the plane tangent to the sphere at the south pole. The chosen face is now the infinite face.