

Fig. 13.3

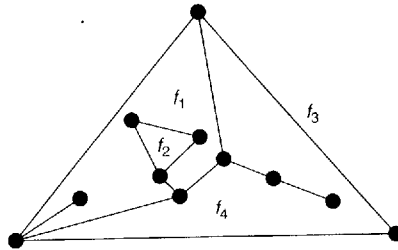


Fig. 13.4

Figure 13.4 shows a representation of the graph of Fig. 13.2 in which the infinite face is f_3 .

We now state and prove **Euler's formula** that tells us that whatever plane drawing of a planar graph we take, the number of faces is always the same and is given by a simple formula. An alternative proof is outlined in Exercise 13.11.

THEOREM 13.1 (Euler, 1750). *Let G be a plane drawing of a connected planar graph, and let n , m and f denote respectively the number of vertices, edges and faces of G . Then*

$$n - m + f = 2.$$

Remark. An example of this theorem is given by Fig. 13.2, where $n = 11$, $m = 13$, $f = 4$, and $n - m + f = 11 - 13 + 4 = 2$.

Proof. The proof is by induction on the number of edges of G . If $m = 0$, then $n = 1$ (since G is connected) and $f = 1$ (the infinite face). The theorem is therefore true in this case.

Now suppose that the theorem holds for all graphs with at most $m - 1$ edges, and let G be a graph with m edges. If G is a tree, then $m = n - 1$ and $f = 1$, so that $n - m + f = 2$, as required. If G is not a tree, let e be an edge in some cycle of G . Then $G - e$ is a connected plane graph with n vertices, $m - 1$ edges, and $f - 1$ faces, so that $n - (m - 1) + (f - 1) = 2$, by the induction hypothesis. It follows that $n - m + f = 2$, as required. //

This result is often called 'Euler's polyhedron formula', since it relates the numbers of vertices, edges and faces of a convex polyhedron. For example, for a cube we have $n = 8$, $m = 12$, $f = 6$, and $n - m + f = 8 - 12 + 6 = 2$ (see Fig. 13.5).

Do disconnected version of 13.1 to prepare for $n = n_1 + n_2$ - type exercises

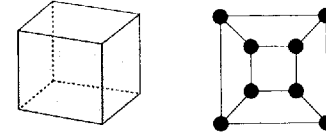


Fig. 13.5

To see the connection in general, project the polyhedron out onto its circumsphere, and then use stereographic projection (as in Fig. 13.3) to project it down onto the plane. The resulting graph is a 3-connected plane graph in which each face is bounded by a polygon – such a graph is called a **polyhedral graph** (see Fig. 13.1). For convenience, we restate Theorem 13.1 for such graphs.

COROLLARY 13.2. *Let G be a polyhedral graph. Then, with the above notation,*

$$n - m + f = 2.$$

Euler's formula can easily be extended to disconnected graphs:

COROLLARY 13.3. *Let G be a plane graph with n vertices, m edges, f faces and k components. Then*

$$n - m + f = k + 1.$$

Proof. The result follows on applying Euler's formula to each component separately, remembering not to count the infinite face more than once. //

All the results mentioned so far in this section apply to arbitrary plane graphs. We now restrict ourselves to simple graphs.

COROLLARY 13.4. (i) *If G is a connected simple planar graph with n (≥ 3) vertices and m edges, then $m \leq 3n - 6$.*
 (ii) *If, in addition, G has no triangles, then $m \leq 2n - 4$.*

Proof. (i) We can assume that we have a plane drawing of G . Since each face is bounded by at least three edges, it follows on counting up the edges around each face that $3f \leq 2m$; the factor 2 appears since each edge bounds two faces. We obtain the required result by combining this inequality with Euler's formula.

(ii) This part follows in a similar way, except that the inequality $3f \leq 2m$ is replaced by $4f \leq 2m$. //

Using this corollary, we can give an alternative proof of Theorem 12.1.

COROLLARY 13.5. *K_5 and $K_{3,3}$ are non-planar.*

Proof. If K_5 is planar then, applying Corollary 13.4(i), we obtain $10 < 9$, which is a contradiction. If $K_{3,3}$ is planar then, applying Corollary 13.4(ii), we obtain $9 < 8$, which is also a contradiction. //

We use a similar argument to prove the following theorem, which will be useful when we study the colouring of graphs.

THEOREM 13.6. *Every simple planar graph contains a vertex of degree at most 5.*

Proof. Without loss of generality we can assume the graph to be connected, and to have at least three vertices. If each vertex has degree at least 6, then, with the above notation, we have $6n \leq 2m$, and so $3n \leq m$. It follows immediately from Corollary 13.4(i) that $3n \leq 3n - 6$, which is a contradiction. //

We conclude this section with a few remarks on the 'thickness' of a graph. In electrical engineering, parts of networks are sometimes printed on one side of a non-conducting plate, and are called 'printed circuits'. Since the wires are not insulated, they must not cross and the corresponding graphs must be planar (see Fig. 13.6).

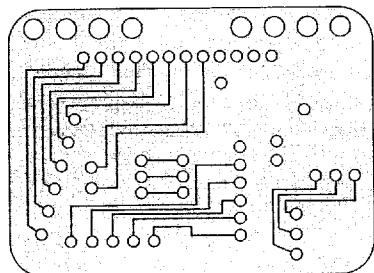


Fig. 13.6

For a general network, we may need to know how many printed circuits are needed to complete the entire network. To this end, we define the **thickness** $t(G)$ of a graph G to be the smallest number of planar graphs that can be superimposed to form G . Like the crossing number, the thickness is a measure of how 'unplanar' a graph is; for example, the thickness of a planar graph is 1, and of K_5 and $K_{3,3}$ is 2. Figure 13.7 shows that the thickness of K_6 is 2.

As we shall see, a lower bound for the thickness of a graph is easily obtained from Euler's formula. Surprisingly, this trivial lower bound frequently turns out to be the

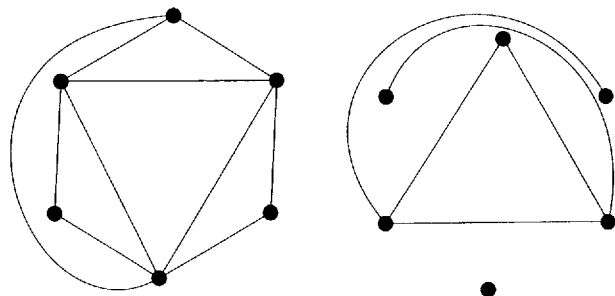


Fig. 13.7

correct value, as we can verify by direct construction. In deriving this lower bound, we use the symbols $\lfloor x \rfloor$ and $\lceil x \rceil$ to denote respectively the largest integer not greater than x and the smallest integer not less than x ; for example, $\lfloor 3 \rfloor = \lceil 3 \rceil = 3$; $\lfloor \pi \rfloor = 3$; $\lceil \pi \rceil = 4$.

THEOREM 13.7. *Let G be a simple graph with n (≥ 3) vertices and m edges. Then the thickness $t(G)$ of G satisfies the inequalities*

$$t(G) \geq \lceil m/(3n-6) \rceil \text{ and } t(G) \geq \lfloor (m+3n-7)/(3n-6) \rfloor.$$

Proof. The first part is an immediate application of Corollary 13.4(i), the brackets arising from the fact that the thickness must be an integer. The second part follows from the first by using the easily proved relation $\lceil a/b \rceil = \lfloor (a+b-1)/b \rfloor$, where a and b are positive integers. //

Exercises 13

- 13.1^s Verify Euler's formula for
- the wheel W_8 ;
 - the graph of the octahedron;
 - the graph of Fig. 13.1;
 - the complete bipartite graph $K_{2,7}$.
- 13.2 Redraw the graph of Fig. 13.2 with
- f_1 as the infinite face;
 - f_2 as the infinite face.
- 13.3^s
- Use Euler's formula to prove that, if G is a connected planar graph of girth 5 then, with the above notation, $m \leq 5(n-2)/3$. Deduce that the Petersen graph is non-planar.
 - Obtain an inequality, generalizing that in part (i), for connected planar graphs of girth r .
- 13.4 Let G be a polyhedron (or polyhedral graph), each of whose faces is bounded by a pentagon or a hexagon.
- Use Euler's formula to show that G must have at least 12 pentagonal faces.
 - Prove, in addition, that if there are exactly three faces meeting at each vertex, then G has exactly 12 pentagonal faces.
- 13.5 Let G be a simple plane graph with fewer than 12 faces, in which each vertex has degree at least 3.
- Use Euler's formula to prove that G has a face bounded by at most four edges.
 - Give an example to show that the result of part (i) is false if G has 12 faces.
- 13.6^s
- Let G be a simple connected cubic plane graph, and let ϕ_k be the number of k -sided faces. By counting the number of vertices and edges of G , prove that $3\phi_3 + 2\phi_4 + \phi_5 - \phi_7 - 2\phi_8 - \dots = 12$.
 - Deduce that G has at least one face bounded by at most five edges.
- 13.7 Let G be a simple graph with at least 11 vertices, and let \bar{G} be its complement.
- Prove that G and \bar{G} cannot both be planar.
(In fact, a similar result holds if 11 is replaced by 9.)
 - Find a graph G with 8 vertices such that G and \bar{G} are both planar.

- 13.8* Find the thickness of
- the Petersen graph;
 - the 4-cube Q_4 .
- 13.9
- Show that the thickness of K_n satisfies $t(K_n) \geq \lfloor (n+7)/6 \rfloor$.
 - Use the results of Exercise 13.7 to prove that equality holds if $n = 8$, but not if $n = 9$ or 10.
- (In fact, equality holds for all n other than 9 or 10.)
- 13.10
- Use Corollary 13.4(ii) to prove that $t(K_{r,s}) \geq \lceil rs/(2r+2s-4) \rceil$, and verify that equality holds for $t(K_{3,3})$.
 - Given that r is even, show that $t(K_{r,s}) = r$, and deduce from part (i) that $t(K_{r,s}) = r/2$ if $s > (r-2)^2/2$.
- 13.11* Let G be a polyhedral graph and let W be the cycle subspace of G .
- Show that the polygons bounding the finite faces of G form a basis for W .
 - Deduce Corollary 13.2.

14 Graphs on other surfaces

In the previous two sections we considered graphs drawn in the plane or (equivalently) on the surface of a sphere. We now consider drawing graphs on other surfaces – for example, the torus. It is easy to see that K_5 and $K_{3,3}$ can be drawn without crossings on the surface of a torus (see Fig. 14.1), and it is natural to ask whether there are analogues of Euler's formula and Kuratowski's theorem for graphs drawn on such surfaces.

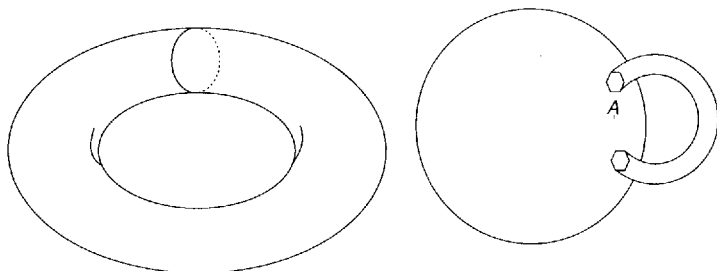


Fig. 14.1

The torus can be thought of as a sphere with one 'handle'. More generally, a surface is of **genus g** if it is topologically homeomorphic to a sphere with g handles. If you are unfamiliar with these terms, just think of graphs drawn on the surface of a doughnut with g holes in it. Thus the genus of a sphere is 0, and that of a torus is 1.

A graph that can be drawn without crossings on a surface of genus g , but not on one of genus $g-1$, is a **graph of genus g** . Thus, K_5 and $K_{3,3}$ are graphs of genus 1 (also called **toroidal graphs**).

The following result gives us an upper bound for the genus of a graph.

THEOREM 14.1. *The genus of a graph does not exceed the crossing number.*

Proof. We draw the graph on the surface of a sphere so that the number of crossings is as small as possible, and is therefore equal to the crossing number c . At each crossing, we construct a 'bridge' (as in Fig. 1.1) and run one edge over the bridge and the other under it. Since each bridge can be thought of as a handle, we have drawn the graph on the surface of a sphere with c handles. It follows that the genus does not exceed c . //

There is currently no complete analogue of Kuratowski's theorem for surfaces of genus g , although N. Robertson and P. Seymour have proved that there exists a finite collection of 'forbidden' subgraphs of genus g , for each value of g , corresponding to the forbidden subgraphs K_5 and $K_{3,3}$ for graphs of genus 0.

In the case of Euler's formula we are more fortunate, since there is a natural generalization for graphs of genus g . In this generalization, a **face** of a graph of genus g is defined in the obvious way.

THEOREM 14.2. *Let G be a connected graph of genus g with n vertices, m edges and f faces. Then $n - m + f = 2 - 2g$.*

Sketch of proof. We outline the main steps in the proof, omitting the details.

Without loss of generality, we may assume that G is drawn on the surface of a sphere with g handles. We can also assume that the curves A at which the handles meet the sphere are cycles of G , by shrinking those cycles that contain these curves in their interior.

We next disconnect each handle at one end, in such a way that the handle has a free end E and the sphere has a corresponding hole H . We may assume that the cycle corresponding to the end of the handle appears at both the free end E and at the other end, since the additional vertices and edges required for this exactly balance each other, leaving $n - m + f$ unchanged (see Fig. 14.2).

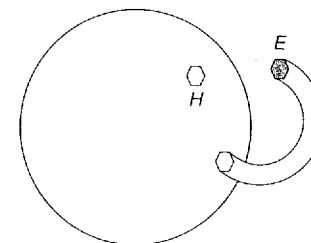


Fig. 14.2

We complete the proof by telescoping each of these handles, leaving a sphere with $2g$ holes in it. This telescoping process does not change the value of $n - m + f$. But for a sphere, $n - m + f = 2$, and hence for a sphere with $2g$ holes in it, $n - m + f = 2 - 2g$. The result follows immediately. //

COROLLARY 14.3. *The genus $g(G)$ of a simple graph G with n (≥ 4) vertices and m edges satisfies the inequality*

$$g(G) \geq \lceil (m - 3n)/6 + 1 \rceil.$$

Proof. Since each face is bounded by at least three edges, we have (as in the proof of Corollary 13.4(i)) $3f \leq 2m$. The result follows on substituting this inequality into Theorem 14.2, and using the fact that the genus of a graph is an integer. //

Just as for the thickness of a graph, little is known about the genus of an arbitrary graph. The usual approach is to use Corollary 14.3 to obtain a lower bound for the genus, and then to try to obtain the required drawing by direct construction.

One case of particular historical importance is that of the genus of the complete graphs. Corollary 14.3 tells us that the genus of K_n satisfies

$$g(K_n) \geq \lceil \{n(n-1)/2 - 3n\}/6 + 1 \rceil$$

or, after a little algebraic manipulation,

$$g(K_n) \geq \lfloor (n-3)(n-4)/12 \rfloor.$$

Percy Heawood asserted in 1890 that the inequality just obtained is an equality, and this was proved in 1968 by Ringel and Youngs after a long and difficult struggle.

THEOREM 14.4 (Ringel and Youngs, 1968). $g(K_n) = \lfloor (n-3)(n-4)/12 \rfloor$.

Remark. We do not prove this here; see Ringel [35] for a discussion and proof of this theorem.

Further results concerning the drawing of graphs on these surfaces, as well as on 'non-orientable' surfaces (such as the projective plane and the Möbius strip), can be found in Beineke and Wilson [27] or Gross and Tucker [29].

Exercises 14

- 14.1^s The surface of a torus can be regarded as a rectangle in which opposite edges are identified (see Fig. 14.3). Using this representation, find drawings of K_5 and $K_{3,3}$ on the torus.

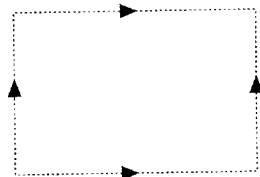


Fig. 14.3

- 14.2 Using the representation of Exercise 14.1, show that the Petersen graph has genus 1.
- 14.3^s (i) Calculate $g(K_7)$ and $g(K_{11})$.
(ii) Give an example of a complete graph of genus 2.
- 14.4 (i) Use Theorem 14.4 to prove that there is no value of n for which $g(K_n) = 7$.
(ii) What is the next integer that is not the genus of any complete graph?
- 14.5^s (i) Give an example of a plane graph that is regular of degree 4 and each face of which is a triangle.
(ii) Show that there is no graph of genus $g \geq 1$ with these properties.
- 14.6 (i) Obtain a lower bound, analogous to that of Corollary 14.3, for a graph containing no triangles.
(ii) Deduce that $g(K_{r,s}) \geq \lceil (r-2)(s-2)/4 \rceil$. (Ringel has shown that this is an equality.)

- 14.7* (i) Let G be a non-planar graph that can be drawn on a Möbius strip. Prove that, with the usual notation, $n - m + f = 1$.
(ii) Show how K_5 and $K_{4,3}$ can be drawn on the surface of a Möbius strip.

15 Dual graphs

In Theorems 12.2 and 12.3 we gave necessary and sufficient conditions for a graph to be planar – namely, that it contains no subgraph homeomorphic or contractible to K_5 or $K_{3,3}$. We now discuss conditions of a different kind, involving the concept of duality.

Given a plane drawing of a planar graph G , we construct another graph G^* , called the **(geometric) dual** of G . The construction is in two stages:

- (i) inside each face f of G we choose a point v^* – these points are the vertices of G^* ;
(ii) corresponding to each edge e of G we draw a line e^* that crosses e (but no other edge of G), and joins the vertices v^* in the faces f adjoining e – these lines are the edges of G^* .

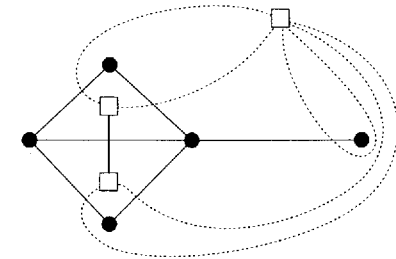


Fig. 15.1

This procedure is illustrated in Fig. 15.1. The vertices v^* of G^* are represented by small squares, the edges e of G by solid lines, and the edges e^* of G^* by dotted lines. Note that an end-vertex or a bridge of G gives rise to a loop of G^* , and that if two faces of G have more than one edge in common, then G^* has multiple edges.

The geometric idea of duality is very old. For example, the 'fifteenth book of Euclid', written about AD 500–600, remarks that the dual of a cube is an octahedron, and that the dual of a dodecahedron is an icosahedron (see Exercise 15.2). Note that any two graphs formed from G in this way must be isomorphic; this is why we called G^* 'the dual of G ' instead of 'a dual of G '. On the other hand, if G is isomorphic to H , it does not necessarily follow that G^* is isomorphic to H^* ; an example that demonstrates this is given in Exercise 15.5.

If G is both plane and connected, then G^* is plane and connected, and there are simple relations between the numbers of vertices, edges and faces of G and G^* .

LEMMA 15.1. *Let G be a plane connected graph with n vertices, m edges and f faces, and let its geometric dual G^* have n^* vertices, m^* edges and f^* faces. Then $n^* = f$, $m^* = m$ and $f^* = n$.*

Proof. The first two relations are direct consequences of the definition of G^* . The third relation follows on substituting these two relations into Euler's formula, applied to both G and G^* . //

Since the dual G^* of a plane graph G is also a plane graph, we can repeat the above construction to form the dual G^{**} of G^* . If G is connected, then the relationship between G^{**} and G is particularly simple, as we now show.

THEOREM 15.2. *If G is a plane connected graph, then G^{**} is isomorphic to G .*

Proof. The result follows immediately, since the construction that gives rise to G^* from G can be reversed to give G from G^* ; for example, in Fig. 15.1 the graph G is the dual of the graph G^* . We need to check only that a face of G^* cannot contain more than one vertex of G (it certainly contains at least one), and this follows immediately from the relations $n^{**} = f^* = n$, where n^{**} is the number of vertices of G^{**} . //

If G is a planar graph, then a dual of G can be defined by taking any plane drawing and forming its geometric dual, but uniqueness does not always hold. Since duals have been defined only for planar graphs, it is trivially true to say that a graph is planar if and only if it has a dual. On the other hand, we cannot tell from the above whether a given graph is planar. It is obviously desirable to find a definition of duality that generalizes the geometric dual and enables us in principle to determine whether a given graph is planar. One such definition exploits the relationship under duality between the cycles and cutsets of a planar graph G . We now describe this relationship and use it to obtain the definition we seek. An alternative definition is given in Exercise 15.11.

THEOREM 15.3. *Let G be a planar graph and let G^* be a geometric dual of G . Then a set of edges in G forms a cycle in G if and only if the corresponding set of edges of G^* forms a cutset in G^* .*

Proof. We can assume that G is a connected plane graph. If C is a cycle in G , then C encloses one or more finite faces of G , and thus contains in its interior a non-empty set S of vertices of G^* . It follows immediately that those edges of G^* that cross the edges of C form a cutset of G^* whose removal disconnects G^* into two subgraphs, one with vertex set S and the other containing those vertices that do not lie in S (see Fig. 15.2). The converse implication is similar, and is omitted. //

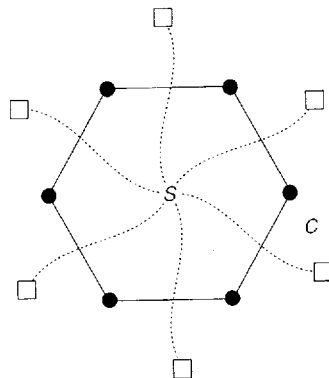


Fig. 15.2

COROLLARY 15.4. *A set of edges of G forms a cutset in G if and only if the corresponding set of edges of G^* forms a cycle in G^* .*

Proof. The result follows on applying Theorem 15.3 to G^* and using Theorem 15.2. //

Using Theorem 15.3 as motivation, we can now give an abstract definition of duality. Note that this definition does not invoke any special properties of planar graphs, but concerns only the relationship between two graphs.

We say that a graph G^* is an **abstract dual** of a graph G if there is a one-to-one correspondence between the edges of G and those of G^* , with the property that a set of edges of G forms a cycle in G if and only if the corresponding set of edges of G^* forms a cutset in G^* . For example, Fig. 15.3 shows a graph and its abstract dual, with corresponding edges sharing the same letter.

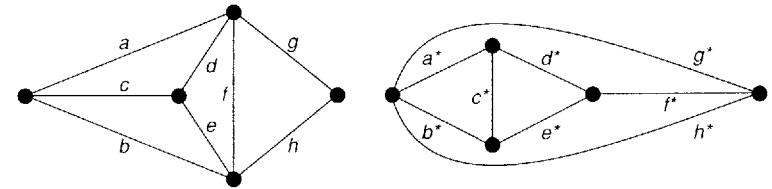


Fig. 15.3

It follows from Theorem 15.3 that the concept of an abstract dual generalizes that of a geometric dual, in the sense that if G is a plane graph and G^* is a geometric dual of G , then G^* is an abstract dual of G . We should like to obtain analogues for abstract duals of some results on geometric duals. We present just one of these here – the analogue for abstract duals of Theorem 15.2.

THEOREM 15.5. *If G^* is an abstract dual of G , then G is an abstract dual of G^* .*

Remark. Note that we do not require that G should be connected.

Proof. Let C be a cutset of G and let C^* denote the corresponding set of edges of G^* . We show that C^* is a cycle of G^* . By the first part of Exercise 5.12, C has an even number of edges in common with any cycle of G , and so C^* has an even number of edges in common with any cycle of G^* . It follows from the second part of Exercise 5.12 that C^* is either a cycle in G^* or an edge-disjoint union of at least two cycles. But the second possibility cannot occur, since we can show similarly that cycles in C^* correspond to edge-disjoint unions of cutsets in G , and so C would be an edge-disjoint union of at least two cutsets, rather than a single cutset. //

Although the definition of an abstract dual may seem strange, it turns out to have the properties required of it. We saw in Theorem 15.3 that a planar graph has an abstract dual (for example, any geometric dual). We now show that the converse result is true – that any graph with an abstract dual must be planar. This gives us an abstract definition of duality that generalizes the geometric dual and characterizes planar graphs. It turns out that the definition of an abstract dual is a natural consequence of the study of duality in matroid theory (see Section 32).

THEOREM 15.6. *A graph is planar if and only if it has an abstract dual.*

Remark. There are several proofs of this result. We outline a proof that uses Kuratowski's theorem.

Sketch of proof. As mentioned above, it is sufficient to prove that if G is a graph with an abstract dual G^* , then G is planar. The proof is in four steps.

- (i) We note first that if an edge e is removed from G , then the abstract dual of the remaining graph may be obtained from G^* by contracting the corresponding edge e^* . On repeating this procedure, we deduce that, if G has an abstract dual, then so does any subgraph of G .
- (ii) We next observe that if G has an abstract dual, and G' is homeomorphic to G , then G' also has an abstract dual. This follows from the fact that the insertion or removal in G of a vertex of degree 2 results in the addition or deletion of a 'multiple edge' in G^* .
- (iii) The third step is to show that neither K_5 nor $K_{3,3}$ has an abstract dual. If G^* is a dual of $K_{3,3}$, then since $K_{3,3}$ contains only cycles of length 4 or 6 and no cutsets with two edges, G^* contains no multiple edges and each vertex of G^* has degree at least 4. Hence G^* must have at least five vertices, and thus at least $(5 \times 4)/2 = 10$ edges, which is a contradiction. The argument for K_5 is similar, and is omitted.
- (iv) Suppose, now, that G is a non-planar graph with an abstract dual G^* . Then, by Kuratowski's theorem, G has a subgraph H homeomorphic to K_5 or $K_{3,3}$. It follows from (i) and (ii) that H , and hence also K_5 or $K_{3,3}$, must have an abstract dual, contradicting (iii). //

Exercises 15

15.1^s Find the duals of the graphs in Fig. 15.4 and verify Lemma 15.1 for these.

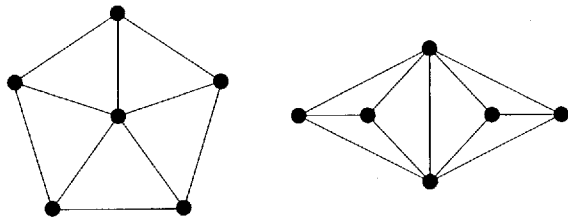


Fig. 15.4

- 15.2 Show that the dual of the cube graph is the octahedron graph, and that the dual of the dodecahedron graph is the icosahedron graph.
- 15.3 Show that the dual of a wheel is a wheel.
- 15.4^s Use duality to prove that there exists no plane graph with five faces, each pair of which share an edge in common.
- 15.5^s Show that the graphs in Fig. 15.5 are isomorphic, but that their geometric duals are not isomorphic.
- 15.6 (i) Give an example to show that, if G is a disconnected plane graph, then G^{**} is not isomorphic to G .
(ii) Prove the result of part (i) in general.

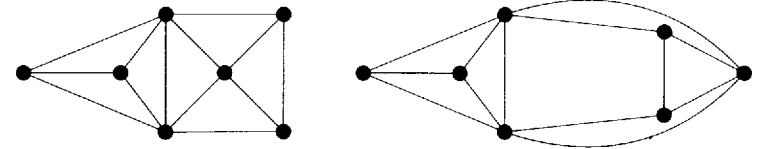


Fig. 15.5

- 15.7^s Dualize the results of Exercise 13.4.
- 15.8^s Prove that, if G is a 3-connected plane graph, then its geometric dual is a simple graph.
- 15.9^s Let G be a connected plane graph. Using Theorem 5.1 and Corollary 6.3, prove that G is bipartite if and only if its dual G^* is Eulerian.
- 15.10 (i) Give an example to show that, if G is a connected plane graph, then any spanning tree in G corresponds to the complement of a spanning tree in G^* .
(ii) Prove the result of part (i) in general.
- 15.11* A graph G^* is a **Whitney dual** of G if there is a one-one correspondence between $E(G)$ and $E(G^*)$ such that, for each subgraph H of G with $V(H) = V(G)$, the corresponding subgraph H^* of G^* satisfies

$$\gamma(H) + \xi(\bar{H}^*) = \xi(G^*),$$
 where \bar{H}^* is obtained from G^* by deleting the edges of H^* , and γ and ξ are defined as in Section 9.
 - (i) Show that this generalizes the idea of a geometric dual.
 - (ii) Prove that, if G^* is a Whitney dual of G , then G is a Whitney dual of G^* .
(In 1932, H. Whitney proved that a graph is planar if and only if it has such a dual.)

16 Infinite graphs

In this section we show how some of the definitions in previous sections can be extended to infinite graphs. An **infinite graph** G consists of an infinite set $V(G)$ of elements called **vertices**, and an infinite family $E(G)$ of unordered pairs of elements of $V(G)$ called **edges**. If $V(G)$ and $E(G)$ are both countably infinite, then G is a **countable graph**. We exclude the possibility of $V(G)$ being infinite but $E(G)$ finite, as such objects are merely finite graphs together with infinitely many isolated vertices, or of $E(G)$ being infinite but $V(G)$ finite, as such objects are essentially finite graphs but with infinitely many loops or multiple edges.

Many of our earlier definitions ('adjacent', 'incident', 'isomorphic', 'subgraph', 'connected', 'planar', etc.) extend immediately to infinite graphs. The **degree** of a vertex v of an infinite graph is the cardinality of the set of edges incident to v , and may be finite or infinite. An infinite graph, each of whose vertices has finite degree, is **locally finite**; two important examples are the infinite square lattice and the infinite triangular lattice, shown in Figs. 16.1 and 16.2. We similarly define a **locally countable** infinite graph to be one in which each vertex has countable degree.

We can now prove the following simple, but fundamental, result.

THEOREM 16.1. *Every connected locally countable infinite graph is a countable graph.*