Preparing for Fractions, Discussion Paper from 2006 PCMI Workshop

1. Introduction

Middle school students need to be prepared to learn fractions through work in elementary school that lays a solid foundation for fractions in the spirit of how addition lays a foundation for multiplication. This essay offers suggestions for early-grade mathematics instruction that provides a better preparation for fractions. A critical component of such instruction is working with units, a topic that is also important in applications of whole number arithmetic.

Fractions have assumed a central role in the workplace. Whether on production lines or managers' desks, many of the numbers one encounters in business today are percents and rates—error rate, interest rate, employment rate, productivity level, etc. Thus all citizens today need to know how to use and interpret fractions. Whole number arithmetic, which once was all the mathematics used in most jobs, is now performed in the workplace by machines for the sake of record keeping as well as accuracy. Whole number arithmetic is still needed for simple mental calculations throughout daily life, but increasingly its primary importance is as the mathematical foundation for future mathematical learning. The next major mathematical topic after whole number arithmetic is fractions. International comparisons like TIMSS reveal that too many U.S. students have trouble making the transition from whole number arithmetic to fractions. Thus, it is natural that mathematics in early grades should be taught with greater attention to preparing students for fractions.

We believe that a solid preparation for fractions also gives a solid preparation for algebra and all later mathematics. Moreover, there is wide agreement that students who do not master fractions—what a fraction is, how to calculate with fractions and apply them—face a tremendous hurdle in trying to master algebra or subsequent college mathematics.

A number of mathematics educators as well as several mathematicians have given considerable thought to preparation for fractions. This essay builds heavily on their efforts, especially those of the Rational Number Project participants (see www.education.umn.edu/rationalnumberproject), Susan Lamon (e.g., [Lamon, 2006]), Les Steffe (e.g., [Steffe et al., 1988]) and H H Wu ([Wu, 2002]). It also draws the development of fractions in Singapore and Japan elementary mathematics curricula. This essay grew out of discussions at a workshop on fractions at the Park City Mathematics Institute in July 2006. The participants consisted of mathematicians, mathematics educators, and mathematics teachers.

2. Moving from Whole Numbers to Fractions

Children enter first grade with an intuitive understanding of whole numbers in the context of counting a collection of objects. Because fractions and the arithmetic of fractions are much more complicated, intuition cannot be counted to develop an understanding in young students' minds of what fractions are, much less how to calculate with them. Notation and terminology are much more important with fractions, but they can cause more problems than they solve.

For students of all ages, definitions of basic mathematical concepts have to be framed with care: not too formal, not too informal. The New Math movement gave rigorous definitions starting with whole numbers, i.e., 5 is the collection of all sets that contain 5 elements, that were out of touch with children's and their parents' mathematical experience. The common sense notion of a whole number as a counting number, to count how many items are in a collection, provides an adequate informal definition in early grades. Later, in anticipation of rational and real numbers, whole numbers can be identified with appropriate points on the number line.

On the other hand, most naïve approaches to understanding of a fraction, such as 1/3, can lead to misperceptions. Thinking of 1/3 in terms of a circle split into three thirds, i.e., three equal pieces whose union is the whole circle, is dangerously narrow. A person with this image of 1/3 might forget that the pieces need to be equal and think of 1/3 as the name of one of the pieces when a circle split into 3 unequal pieces.

As soon as students are ready, a fraction should be defined as an appropriate point on the number line. However, there is another definition of fractions that is more useful for calculations, that is used in many other countries:

A fraction is a number that is an integral multiple of some unit fraction.

In mathematical notation, we mean a number of the form $k(1/l)$, for whole numbers k, l ($l > 0$). This definition assumes that students have first developed a good understanding of what a unit fraction is. Unit fractions are discussed extensively in the next section.

Well before students study fractions formally, they encounter fractions in a variety of everyday situations, some studied in school—telling time, making change, cooking recipes, sharing (when portions are not whole amounts), and measuring small lengths. The Japanese elementary school teacher's guide lists five different interpretations of fractions [Akihiko et al., 2006]. When fractions are first studied in third grade in Japan, the following two interpretations are given for a fraction, such as 2/3:

1. Representing two of three equally divided parts.
2. Representing the quantity resulting from a measurement, such as 2/3 meter.

Note that the first interpretation is based on an unspecified whole. While Japanese teachers relish mathematical situations that are open to multiple interpretations, many U.S. elementary teachers are accustomed to teaching just one way to interpret or solve problems. Furthermore, there is the danger that the different interpretations of fractions may be viewed by students and teachers as equivalent definitions of fractions, heightening confusion about what is a fraction.

Defining fractions in terms of unit fractions avoids a major conceptual problem, namely, establishing that a fraction is a number. That burden now falls to unit fractions. A second advantage of defining fractions in terms of unit fractions is that this approach separates the study of the numerator and the denominator of a fraction. Numerators are standard counting numbers, while denominators are a totally new quantity—they are units defined in terms of reciprocals. Again, this is the reason we focus in the rest of the article on unit fractions and units generally.

With so much time in early grades devoted to whole number arithmetic, students unconsciously reinforce their initial intuition that the term 'number' means only a 'whole number' or 'counting number.' A U.S. fourth grade student who has encountered fractions in measurement (time, money, lengths, etc.) and other contexts will still likely say that a fraction is not a number, but rather is a part of something. The Rational Number Project devoted considerable effort to understanding the hurdles to learning fractions created by students' belief that 'number' = 'whole number.' A similar problem arises with multiplication, which is initially learned as repeated addition, i.e., multiplication by a whole number. In this context, multiplication by a fraction makes no sense.

Because so many U.S. students never move beyond thinking of a number as a counting number, we should not be surprised that students mindlessly memorize operations with fractions in terms of the integers in numerators and denominators without knowing what a fraction is or that they will assert that 1/3 + 1/5 = 1/8.

On the other hand, a child's first understanding of a number will necessarily be as a counting number and multiplication is naturally introduced as repeated addition. Thus the pedagogical goal must be to help students extend, rather than abandon, these initial understandings of a number and multiplication; Les Steffe calls this critical process reconceptualization. Students face this challenge over and over as they advance in school mathematics.

Students do develop a valid understanding of fractions as numbers in practical settings. For example, the students know that one fourth of a particular item (e.g., of a pie or a quart) plus two fourths of that item equals three fourths of the item. Further, these students can often find natural common units for adding fractions in familiar contexts, e.g., half an hour plus a third of an hour equals 50 minutes.

The preparation for fractions can build on students' early experience with fractions as parts of something and their readiness to do simple calculation with fractions. However, it is a very big conceptual step to go from thinking about fractions as parts of given objects, such as pies, to thinking about fractions as potential parts of an unspecified object.

We close this section by mentioning the ambiguity in fractional notation. The expression a/b, where a, b are whole numbers and b > 0, has (at least) two mathematical meanings. It is a rational number equal to the fraction a(1/b). It is also a common way of writing the calculation a ÷ b. Students need to have a good understanding of fractions (the first interpretation), before the relationship between fractions and division (the second interpretation) is presented. If fractions are presented in the context of division, students can easily think of a fraction, not as number in its own right, but rather as the quotient of whole number division. In this flawed framework, it makes sense to learn arithmetic operations on fractions by memorizing integer-valued formulas for the resulting numerators and denominators.

3. Unit Fractions

Unit fractions, such as ¼, are a natural precursor to fractions. Unit fractions arise frequently in day-to-day conversations—a quarter (the coin), a quarter after 5 o'clock, a quarter of a mile down the road, a quarter of a cup of flour, a 1 ¼ inch screw, etc. A growing number of U.S. mathematics textbooks discuss unit fractions to varying degrees starting in first grade. However, a more carefully planned, long-term program of instruction about unit fractions is needed.

For young children, unit fractions evolve from counting numbers: a pie divided into fourths is split into equal pieces which when counted amount to 4. Given two pies divided into sixths with 3 sixths left in the first pie (the three other sixths were eaten) and 2 sixths left in the second pie, first-grade students can count, and later add, the sixths in the two pies to obtain a total of 5 sixths.

Here is a class activity that Howe recommends for emphasizing the difference between counting numbers and unit fractions. Given a pitcher with a capacity of one quart of water, a student can determine the capacity, say 4, of a second pitcher by counting how many quart-size pitcherfuls it takes to fill up the new pitcher. Now consider a third pitcher whose capacity is a 1/4th of a quart. To determine what this unit-fraction is, the student needs to determine how many pitcherfuls of the third pitcher it takes to fill the quart pitcher.

As noted above, while it natural to use pictures of pies or some other common geometric figure when starting to work with unit fractions, there are several misconceptions that can arise from such geometric examples of unit fractions. Extensive examples with measurement and with dividing a collection of items in equal shares can help students develop a more general mental understanding of unit fractions.

There are a number of different steps that extend this basic start with fractions. We offer one suggestive progression from 1st through 3rd grades, knowing that others are better qualified to address this development. Intermixed with pictorial problems about unit fractions would be an occasional set of purely numerical problems (no figures) involving simple addition of fractions, such as 2/5 + 2/5. Another step is to have problems whose answers are improper fractions and then to restate an answer such as 5/4 cups of sugar as 1¼ cups of sugar. A parallel step is to have answers that are whole numbers, such as 4 fourths or 8 fourths, and to convert from fourths to whole numbers. Likewise, one can do conversions from whole numbers to unit fractions; e.g., if two pies are cut up into fourths and Sarah takes away 3 quarters, how many fourths are left. Simple examples of multiplication and division of amounts stated in unit fractions can be introduced, e.g., given a recipe requiring 2 fourths of a cup of sugar; how many batches of the recipe can we make with two cups of sugar. Initially these problems would be accompanied with diagrams to help organize students' thinking. For example, in the situations above with cups of sugar, there would be figures with the cups divided into fourths with horizontal lines.

In third grade, two-step word problems about unit fractions might be posed (initially with helpful diagrams). For example:

> Three children have two cupcakes. Each of the two cupcakes is cut in fourths. One child takes two fourths. The remaining pieces are split between the other two children. How many fourths does each of those two children get?

When division is learned, one can determine how many balls constitute one fourth of a given set of 20 balls, and later determine what 3/4, i.e. 3 fourths, of the set is.

The point is that extensive arithmetic exercises with unit fractions lead students both to be comfortable solving problems with fractions when posed as integral multiples of unit fractions. To qualify the amount of work with fractions, we suggest 10% as a ballpark percentage of arithmetic problems in early grades that should be posed in terms of unit fractions.

These arithmetic experiences are reinforced and extended by the use of unit fractions in measurement problems. It is important that all whole number arithmetic be interpreted in terms of measuring lengths, e.g., addition is concatenation of lengths; multiplication is repeated concatenation of lengths. Thus the number line has a natural metric interpretation as distances from 0 (the start point). Unit fractions arise naturally in measuring lengths. Unit fractions also have a natural role in the measurement of time, money, and later area and volume. Note that the transition from multiplication by whole numbers, i.e., repeated addition, to multiplication by fractions is a natural extension in linear measurements: if bricks are 8 inches long, how would long would a row of 3½ bricks be?

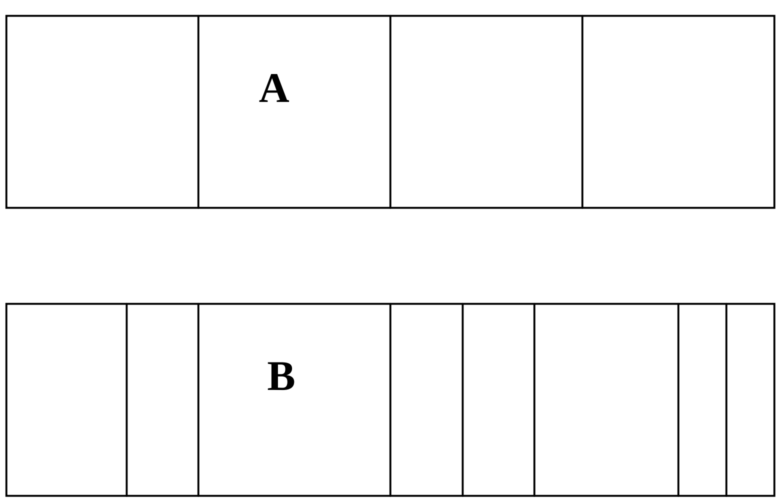
Close to 20% of measurement problems in grades 2-4 might involve fractions in some way. This includes problems like converting a fraction of a foot to inches. The most important of these contexts is length as measured with a ruler, a concrete version of the number line. While standard rulers are subdivided into halves, fourths, eighths, and sometimes sixteenths of an inch (or of a metric unit), students should have access to rulers with different types of subdivisions, e.g., in 5ths and in 10ths of an inch. In measuring lengths, students are initially equating whole numbers with lengths. Over time, it becomes natural to view all lengths as numbers, and an intuitive feeling for numbers as points on the number line develops. This is why Wu likes to define fractions in terms of the number line. Concurrently, one can transition from unit fraction problems with wording like ‘how many fourths of the pie do the two children have together’ to problems with wording such as ‘how much pie do the two children have together.’

As students start formal work with fractions, e.g., equivalent fractions and finding common denominators to fractions, word problems with unit fractions can continue. An example is the problem of how high a pile of 4 notebooks would be if each notebook is 5/8” thick. The answer would first be found in terms of 1/8ths and then converted to whole inches. A more advanced, inverse version of this problem would be, how many notebooks that are 5/8” thick can be piled into a box that is 2½ feet deep.

The details for integrating this unit-fraction program into the curriculum will challenge mathematics education experts. As an example of pedagogical timing, we cite the question, when should work with rulers lead to the number line being introduced. One response to this challenge is the Rationale Number Project’s volume of sample lessons for introducing fractions in grades 4 and 5 [Cramer et al., 1997].

We noted above that unit fractions are derived in students’ minds from counting numbers as follows: a pie divided into fourths is split into equal pieces which count to 4. To give a sense of the cognitive challenge students face in moving beyond this image of unit fractions, we cite a scene from a demonstration class of fifth graders led by D. Ball at the Park City Mathematics Institute in summer 2006. When students were asked to go to the blackboard and highlight 1/8th of a collection of 24 circles that had been drawn, one student first divided 8 into 24 to get 3, and then he proceeded to partition the set of 24 circles into groups of 3. He had to check that 8 groups of 3 balls completely partitioned the set of 24 balls before being able to say that 3 balls were 1/8 of the set of 24 balls. That is, 1/8 of a set did not exist in his thinking independently of the other 7 1/8’s of the set.

Here is another example of the trouble that students have in moving beyond the equal division model [Tzur, 2006]. Consider the following two rectangles, both with the same dimensions. The upper one is divided into 4 equal sections. The lower one is divided into 8 unequal sections. We are told that section A in the upper rectangle is the same size as section B in the lower rectangle. The question is, what fraction of the lower rectangle is section B?



Many middle school students will assert that section B is our fourth of the upper rectangle but that one cannot tell what fraction it is of the lower rectangle. The same response is given by some middle school teachers.

4. Units

To illustrate the role of units in working with fractions, consider the following problem:

Some balls are taken from a box and 15 balls are left. This number 15 is three quarters (*) of the number of balls that started in the box. How many balls started in the box?

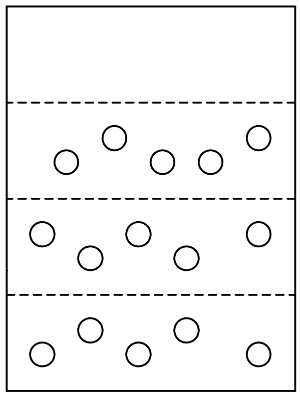
The reasoning for solving this problem involves two types of units. The problem can be restated: if we know 3 fourths of a quantity, what is 4 fourths of the quantity. The key is to think in terms of fourths. If one fourth is our unit, then the problem comes, if three units equal 15, what do four units equal. The natural intermediate step in the solution is to determine what one unit equals. We get that one unit is 15/3 = 5, balls, and the boxful of 4 units equals 4 x 5 = 20 balls.

While fourths were the units for initially analyzing the problem, 5's were the units involved in determining the final answer. One could say that one unit equals our fourth of a boxful, and then restate that unit as equal to 5 balls. One could also look at these two units as a ratio: 5 balls per fourth of a boxful. Analyzing relationships between two or more units underlies the solution of almost all real-world problems involving fractions. This analysis is a natural extension of problems involving one unit, which underlie most simple word problems involving whole number multiplication and division. For example, given that one box holds 5 melons (that is, one unit is 5 melons), one may ask how many melons are in 4 boxes, or ask how many boxes are needed to hold 20 melons. Many educators refer to the (implicit or explicit) use of units to solve such a problem as multiplication reasoning. Such reasoning is a prerequisite fraction problems.

Problem (*) could be modeled algebraically as (3/4)x = 15 and solved for x to obtain x = 15/(3/4), with the right-hand side computed with the invert-and-multiply rule for division by fractions. That rule, of course, yields the same calculation as in the previous analysis: divide 15 by 3 and multiply the result by 4 (or the order could be inverted). We want students to understand the reasoning that leads to the invert-and-multiply rule by working many problems like (*). Then they will think of that rule as simply a mental shortcut for this reasoning.

More generally, when students learn the algorithms for arithmetic operations on fractions, we want the algorithms to be viewed as a concise way of working with previously studied problems involving unit fractions, similar to the same way with whole numbers that the multi-digit multiplication algorithm is seen as a concise way to perform iterated addition.

In East Asian countries, where elementary school children work with unit fractions and solve multi-step word problems, problem (*) is almost routine by 5th grade. We think it is quite realistic to have the same expectations of U.S. 5th graders, if U.S. curricula is laying the proper foundation in early grades. We note that one important advantage for East Asian students is that their textbooks make much more extensive use of diagrams and other helpful figures to allow students to 'see' the right way to look at a problem. (The TIMSS 1999 Video Study [NCES, 2003] found that 83% of the problems in 8th grade math lessons in Japan used diagrams or drawings while the percentage in the U.S. was just 26%.) For example, when problems like (*) are first encountered, students should see a diagram like:



to point them towards the solution. We note that such diagrams would seem to be particularly powerful in helping students organizing their thinking about problems that involve units. On the other hand, students must in time abstract the thinking that is initially motivated by diagrams.

Many rate problems have a similar structure to (*). For example,

> If a car going at a constant speed covers 48 miles in ¾ of an hour, how far will it go in one hour; or equivalently, how fast is it going (in miles per hour).

First we must focus on measuring time in fourths of an hour. Then we switch to the dual unit of 16 miles, the distance traveled in a fourth of an hour.

Of course, problem (*) and the car problem would be studied after simpler problems are worked, such as:

> Find 3/4ths of 20.

Not only is the role of fourths more clear-cut in this new problem, but the intermediate step of determining what one fourth equals involves the standard way of dividing a given amount into fourths by dividing by 4.

Another way to prepare for problem (*) and the associated car problem is to model the same arithmetic calculations with a problem without fractions. For example:

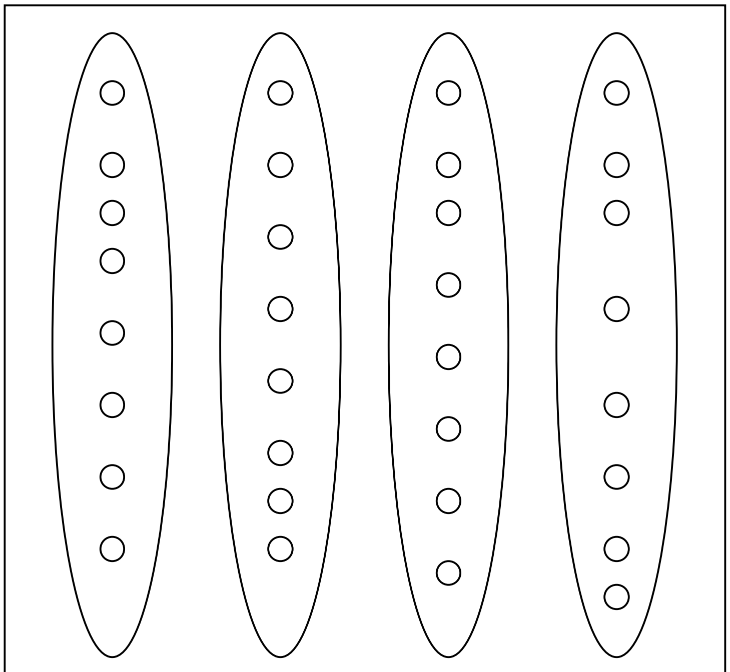
> If 3 boxes hold 15 balls, how many balls will 4 boxes hold? and

> If a bicycle going at a constant speed goes 48 miles in 3 hours, how far will it go in 4 hours?

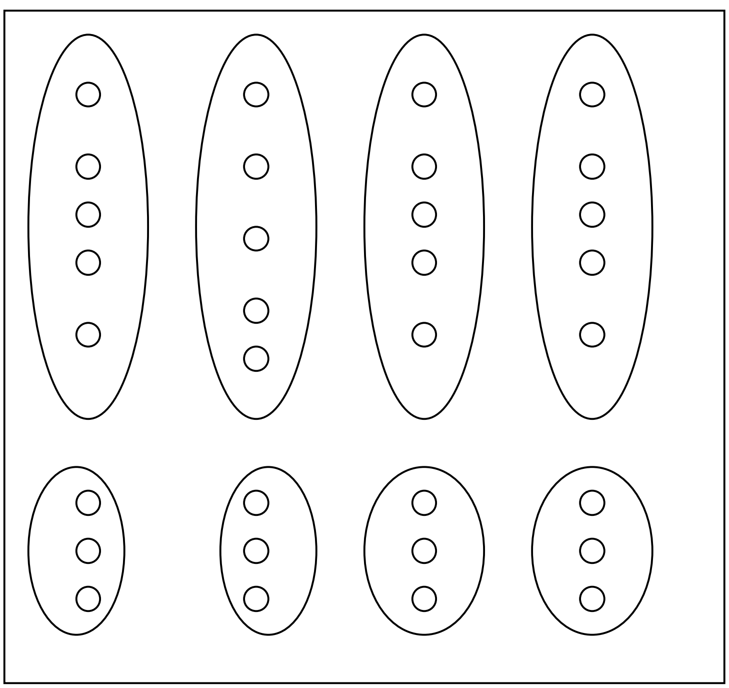
Again, the dual units are much easier to find here: the unit of one box must be equated with the contents of the box, 5 balls; and the unit of one hour equated with the distance traveled in one hour, 16 miles.

If unit fractions are introduced early, all the preceding problems might be worked with whole number arithmetic, before the formal study of fractions and fraction arithmetic, and the study of fractions might need to be delayed a bit in order to devote more time to these unit problems. However, if students can solve the preceding problems, the study of fractions can move faster and still produce greater understanding. Students should be more proficient in applying fractions to proportion, rate and ratio problems, since they will have already learned the reasoning with units that are critical to solving these problems. We make this suggestion to illustrate the types of major rethinking of curricula that would accompany heightened attention to preparation for fractions in elementary school.

Let us briefly touch on the role of units in beginning whole number arithmetic. Skip counting, e.g., 2. 4. 6. 8, .. etc. is counting with a unit larger than 1. Skip counting evolves into multiplication. The distributive law is first introduced in Singapore schools in second grade as a tool for simplifying multiplication with digits larger than 5. The underlying idea is that a large unit in skip counting can be replaced with two smaller units. For example, to determine 4 x 8, that is, the sum of 4 eights, one can start with the diagram



Then we reorganize the groups of 8 balls as groups of 5 balls and 3 balls to get



A student is thus prompted to solve 4 x 8 as 4 x 5 + 4 x 3 (each product obtained by skip counting). This problem is an example of an important topic, changing units, which should run throughout the K-8 curriculum. The next section is devoted to this topic.

We close this section by mentioning a pedagogical strategy, developed by Herb and Ken Gross, for understanding the relationship between numbers and units. They call numbers 'adjectives' and they initially use these 'adjectives' only in the context of modifying a 'noun', such as 5 pencils or 2/3rds of a pie. The nouns are extended to include units of measurement and units defined in terms of other adjective-noun pairs such as 4 (boxes of 500 pencils) and 5 (eighths of an inch).

5. Converting Between Units

One of the critical mathematical building blocks for working with fractions is equivalent fractions, different fractions that represent the same rational number, e.g., ½ or 2/4 or 5/10 or 13/26, etc. Equivalent fractions is often the first topic discussed when the formal study of fractions begins in around 4^{th} grade, after which addition of fractions can be presented. However, the general topic of equivalent representations of a number arises repeatedly in measurement problems, e.g., ½ foot = 6 inches, or 50 cents = 10 nickels = 5 dimes = 2 quarters = ½ dollar, as does the issue of finding a new, common representation for adding quantities in different units, e.g., adding 1/3 foot + ¼ foot by converting to inches, or adding 2 dimes and 1 quarter by converting to cents. Simple cases of equivalent fractions can also be studied: observing on a ruler that half an inch equals 2 fourths of an inch or 4 eighths of an inch, or measuring out 1 1/2 pounds of candy in different ways given bags of candy weighing ¼ lb., 1/8 lb. and 1/10 pound.

Equivalent fractions are a particular case of a more general mathematics topic, namely converting a number expressed in terms of one unit to another unit. Finding a new unit for representing different quantities arises in word problems involving multiplication and division.

Consider the problem:

A brick is 8 inches long. How many bricks must be placed end to end to reach 10 feet?
First we express the length of 10 feet in terms of inches—120 inches— using the conversion rule 1 foot = 12 inches. This is the first change of units. Then we convert the length in inches into another unit, brick lengths, using the conversion rule 1 brick length = 8 inches. The first conversion involved a multiplication and the second a division.

The following solution strategy follows the spirit of unit fraction examples in the previous section. After noting that one brick length is 2/3rds of a foot, one converts the total length from feet to 1/3rds of a foot. This is an easy conversion—multiply by 3-- to keep straight in one's mind, and work with unit fractions continually reinforces such conversion strategies. So now the length is 30 1/3rds of a foot. Since each brick is 2 1/3rds of a foot long, we need 30/2 = 15 bricks.

This problem can be simplified if one knows how to compute with fractions. There is the following condensed solution using a single change of units: 1 brick length = $2/3^{rd}$ of a foot. Now the change of units requires dividing 10 by 2/3. However, the division by 2/3rds can be avoided if we restate the conversion as feet to brick lengths: 1 foot = 1 ½ brick lengths. Then 10 feet = 1 ½ x 10 = 15 brick lengths.

This problem illustrates the fact that any time we perform multiplication or division in solving an applied problem, we are explicitly or implicitly converting units. Thus, more attention to units and change of units in elementary school mathematics instruction seems critical to improving students' mathematical problem-solving skills as well as providing the proper foundation for learning fractions.

As a pedagogical aside, when students are ready to study such a problem, it is very valuable to have a class discussion about different ways to solve the problem (the solution involving division by 2/3 would be too advanced). Looking at multiple ways to solve such problems highlights the role of units and shows how a calculation with a fraction can frequently be recast as a short cut for a two-step calculation involving a multiplication and a division with whole numbers.

Let us next consider a word problem involving three units, a problem more complicated that most U.S. 5th or 6th graders can currently solve. It is the first word problem to appear in the 5th grade Singapore mathematics textbook [Singapore Math, 1997]:

> Mrs. Li bought 420 mangoes for $378. She packed them into packets of 4 mangoes each and sold all the mangoes at $6 per packet. How much money did she make?

The initial units that appear in the problem statement are mangoes and dollars. Later in the problem statement, packets enter. We need to convert units for measuring mangoes from individual mangoes to packets of 4 mangoes. Given that 4 mangoes go into packet, we divide 420 by 4 to obtain 105 packets. Now we convert our units for measuring mangoes from packets to value in dollars. The conversion factor is that one packet yields $6 dollars, and so we multiply for this conversion to obtain a value of 105 x $6 = $630. Note that the conversion factor naturally occurs as the second term in both the multiplication and division conversion step. Finally, we have cost and income in the comparable units, dollars, and so the amount of money made in this activity, $630 - $378, can be computed.

Another way to approach this problem is to look for a way to convert directly from units of mangoes to units of money. This conversion requires determining a rate of income per mango. Since 4 mangoes in a packet sell for $6, we obtain a rate of $6/4 (= $1½ per) per mango.

As students move from elementary school to middle school, greater proficiency in converting between units can be used not only to solve harder word problems but also to gain insight into multiplication and division of fractions. Consider the calculation

2/3 x 4/5

Interpreting 2/3 as 2 thirds [=2(1/3)], we first need to find 1/3 of 4 fifths. We are initially stuck because 1/3 of 4 is not a whole number. We change to new unit that is sure to work, namely 1/(3x5). So we convert 4 fifths to 12 fifteenths [= 12(1/15)]. We can find 1/3 of 12 fifteenths by dividing 12 by 3; it is 4 fifteenths (4/15). Finally we multiply this amount by 2 to find 2 (1/3) = 2 x [4(1/15] = 8/15. Diagrams can help with this problem. For example, 4/5 could initially be depicted with a rectangle partitioned by horizontal lines into 5 equals parts with the lower four parts shadowed. Then the rectangle could be subdivided with 3 vertical lines into 15 equals parts. One third of the 12 shadowed parts is found, etc.

Suppose instead we want to calculate

5/4 ÷ 2/3

It is simplest to use the indirect definition of division to restate this problem as: 2/3rds of what number is 5(1/4). First we must determine how much 1/3 of 5(1/4) is. To do this we divide 5(1/4) by 2, which requires a change of units to restate 5/4 as 10/8. After we know that 1/3 of 5(1/4) is 5(1/8), we can multiply by 3 to get the answer.

Students' knowledge of unit problems can also be used to revisit whole number addition and subtraction from an advanced viewpoint and realize the role of conversion among decimal units in the standard algorithms of arithmetic. The place value notation is now seen as a system of decimal units. The key steps of carrying in addition and borrowing in subtraction involve converting between consecutive decimal units. The standard multiplication and division algorithms can be studied in terms of how they combine partial computations in different decimal units. As an aside, students should find decimal numbers (e.g., 35.26) and order-of-magnitude approximations much easier to understand if they have previously had the extensive study of units that we are advocating.

We conclude this section with an important subtlety about the role of units in division. Interpreting multiplication as repeated addition, the equation 4 x 5 = 20 says that the sum of four 5's is 20. Inverting this process, 20 ÷ 5 = 4 could be interpreted as saying that 4 is the number of 5's that need to be summed to get 20. What then is the interpretation of 20 ÷ 4 = 5 in terms of the multiplication 4 x 5 = 20? It is, what number when summed 4 times yields 20. A more familiar way to state this is, when we divide 20 into 4 equal parts, what is the size of each part. The first problem 20 ÷ 5 = 4 was a change of units: we count numbers by 5's instead of by 1's. The second problem is a partitioning situation, although it can also be interpreted with a change of units as follows: what should the units be if we want to 4 units to equal 20.

People often say that division is the 'inverse' operation of multiplication. However, as just noted, in terms of applied settings, there are two very distinct interpretations of how division is the 'inverse' of multiplication. Students need a program of carefully crafted lessons and exercises over several years to prepare them to work with these two interpretations of division that arise repeatedly in problems involving rates, ratios and proportions. Much research has gone into this pedagogical challenge, e.g., see Rational Number Project website.

6. Concluding Remarks

The search for ways to better prepare U.S. students to learn fractions has led the authors to a greater appreciation of the role of units throughout K-8 instruction. A focus on units can help unify reasoning for solving many word problems with whole numbers as well as with fractions.

This essay tries to sketch the goals and challenges for a program of elementary school mathematics instruction about unit fractions, and units more generally, that can lead into better middle school instruction about fractions and their uses. However, there are more questions raised here than answered when it comes to developing actual textbooks and planning lessons. No matter how good students' preparation for fractions may be, there will still remain major challenges in teaching these students to analyze and solve applications of fractions to proportions, rates, and ratios. This essay has said little about these problems.

Fortunately, many of the details of how young children learn about units and unit fractions and then later learn about fractions and their applications have been the focus of research by mathematics educators. Several mathematicians, most notably H H Wu, have also devoted considerable effort to improving the teaching of fractions. We believe that it is time for mathematicians, mathematics educators and teachers to work together to develop recommendations for improved instruction about fractions, unified across grades K-8, in a fashion that integrates the reasoning with units discussed above.

As more attention is given to reasoning with units, it seems natural that school mathematics curriculum development may involve more educational collaborations with physical scientists and science educators, since units play such an important role in science.

We close with a concrete example of the challenges in implementing the program outlined above. We refer again to the model class of Deborah Ball's mentioned at the end of section 2 where students were asked to find one eighth of 24 balls drawn on the blackboard. One student divided 8 into 24, and, based on his answer of 3, partitioned the 24 balls into 3 groups of 8 each. Next he marked one ball in the first group of 8. However, the student then stopped and gave 1 as the answer. A cognitive specialist watching the students speculated what had gone wrong. Like many other students of his age, this student had trouble keeping track of more than two units at one time. He reorganized the problem of finding $1/8^{th}$ of the whole group of 24 by first breaking 24 into 3 groups (units) of 8's. He then determined what $1/8^{th}$ of a group of 8 was, but had lost track of the relationship between the original group of 24 and the group of 8. Keeping track of multiple units is an example of a critical cognitive skill that mathematicians probably take for granted. Thus, to better prepare students to learn fractions, one needs not only to understand the proper mathematical development of underlying concepts, such as units, but also to understand the hurdles that students face when they try to learn these concepts.

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