European Option Pricing under Geometric Lévy Processes with Proportional Transaction Costs

Haipeng Xing* Yang Yu† Tiong Wee Lim ‡

January 20, 2016

Abstract: Davis et al. (1993) provided a detailed discussion on the problem of European option pricing with transaction costs when the price of underlying follows a diffusion process, we consider a similar problem here and develop a new computational algorithm for the case that the underlying price follows a geometric lévy process. Using an approach that is based on maximization of the expected utility of terminal wealth, we transform the option pricing into stochastic optimal control problems, and argue that the value functions of these problems are the solutions of a free boundary problem, in particular, a partial integro-differential equation, under different boundary conditions. To solve the singular stochastic control problems associated with utility maximization and compute the value function and no-transaction boundaries, we develop a coupled backward induction algorithm that is based on the connection of the free boundary problem to an optimal stopping problem. Numerical examples of option pricing under a double exponential jump diffusion model are also provided.

Keywords: Jump-diffusion processes; Transaction costs; Singular stochastic control

*Department of Applied Mathematics and Statistics, State University of New York, Stony Brook, USA. E-mail:xing@ams.sunysb.edu.
†Department of Applied Mathematics and Statistics, State University of New York, Stony Brook, USA. E-mail: yyu@ams.sunysb.edu.
‡Department of Statistics and Applied Probability, National University of Singapore, Republic of Singapore
1 Introduction

The problem of option valuation has been intensively discussed in quantitative finance ever since the publication of two seminal papers by Black and Scholes (1973) and Merton (1973). In both papers, the authors considered an idealized setting where an investor incurs no transaction costs from trading in a complete market consisting of a risk-free asset and a risky asset whose price is a geometric Brownian motion, and presented a dynamic self-financing trading strategy that replicates the payoff of an option. They then argued that, in the absence of arbitrage, the value of an option equals the amount of initial capital required for setting up the replicating portfolio.

One important assumption in the Black-Scholes-Merton model is the geometric Brownian motion for the stock price process, which has been challenged by a large amount of empirical studies; see Cont (2001), Lai and Xing (2008) and reference therein. To capture asymmetric, leptokurtic features in the underlying asset return distributions and volatility smiles in the option markets, Ait-Sahalia (2002), Carr and Wu (2003), Eraker et al. (2003), Broadie et al. (2007) developed models and methods that incorporate the occurrence of rare jumps in a price process that would otherwise follow a diffusion. Merton (1976) argued that the asset price fluctuations can be decomposed as the sum of “normal” vibrations, caused by temporary imbalance of supply and demand and other new information that causes marginal changes in the stock value, and “abnormal” vibrations, caused by the arrival of important new information which generates occasional and large impact on price. Furthermore, he modeled the normal and abnormal variations by a standard geometric Brownian motion and a Poisson process, respectively, and derived an option pricing formula. Subsequent work has been moving towards considering other or more general Lévy process. For instance, Chan (1999) considered the problem of pricing contingent claims on a stock whose price process follows a geometric Lévy process, and Kou (2002) proposed a double exponential jump-diffusion model and derived analytical solutions for a variety of option-pricing problems.

Another important assumption in the Black-Scholes-Merton model is no transaction costs in the replicating process. Under this and continuous trading assumptions, the option can be perfectly replicated in such a “complete” market. However, in the presence of proportional transaction costs in capital markets, the continuous replication policy incurs an infinite amount of transaction costs over any trading intervals, and perfect hedging becomes impossible, hence the absence of arbitrage argument is no longer valid. To deal with the problem of option pricing and hedging with transaction costs, several
approaches have been proposed for the case that the underlying stock price follows a geometric Brownian motion. One approach is based on super-replication in a discrete-time setting and tries to find trading strategies which produce payoffs at expiration that are larger than or equal to the option payoff; see details in Leland (1985), Boyle and Vorst (1992), Bensaid et al. (1992), Edirisinghe et al. (1993), Soner et al. (1995), and Primbs (2009). Another approach examines the difference between the desired payoff at maturity and the realized cash flow from a hedging strategy, and tries to achieve the best possible tradeoff between the cost of payoff and the risk. A pioneer work along this line is Hodges and Neuberger (1989), who used a risk-averse utility function to assess the replication error and formulated the problem of option pricing and hedging as that of maximizing the investor’s expected utility of terminal wealth. Their key idea behind the utility-based approach is to make use of an indifference argument and to define the writing price of an option as the amount of money that would make an investor indifferent, in terms of expected utility, between trading in the market with and without writing the option. They interpreted the “hedging” of the option as the difference in the two trading strategies, with and without the option. Davis et al. (1993) modified certain settings of Hodges and Neuberger (1989) and developed rigorously the model of Hodges and Neuberger (1989) for a market with proportional transaction costs. In particular, Davis et al. (1993) showed that the option pricing problem with transaction costs involves solving two singular stochastic control problems formulated by Davis and Norman (1990). They also developed, for the negative exponential utility function, numerical algorithms to compute the optimal hedge and option price by making use of discrete-time dynamic programming for an approximating binomial tree for the stock price. Further contributions to the study of the utility based option pricing approach and numerical methods with proportional transaction costs include Clewlow and Hodges (1997), Whalley and Wilmott (1997), Andersen and Damgaard (1999), Constantinides and Zariphopoulou (1999, 2001), Zakamouline (2006), Lai and Lim (2009), and some others.

In this paper, we relax the assumptions of geometric Brownian motion for the stock price process and no transaction costs in the replicating process, and consider the problem of option pricing under general jump diffusion processes with proportional transaction costs by generalizing the utility-based approach developed by Davis et al. (1993) and proposing a coupled backward induction algorithm to compute the solutions. In particular, under the assumption that the underlying stock price follows a geometric Lévy process, we formulate the problem of option pricing with proportional transac-
tion costs as the maximization of the investor’s expected utility of terminal wealth, and
demonstrate that the implied singular stochastic control problem can be reduced to a
free boundary problem for a partial integro-differential equation. The derived equation
is in the form of backward stochastic differential equations, and can be solved by least-
squares Monte Carlo regressions (Gobet et al., 2005; Bender and Steiner, 2012) or Fourier
method (Fang and Oosterlee, 2008; Ruijter and Oosterlee, 2015). Instead of using the
above method, we consider probabilistic numerical methods to solve the derived equa-
tions for the negative exponential utility function. However, our numerical method is
different from Davis et al. (1993), Clewlow and Hodges (1997) and Zakamouline (2006)
who use discrete-time dynamic programming for an approximating binomial tree for
the stock price. Instead, we make use of the connection of the free boundary problem to
an optimal stopping problem, and propose a coupled backward induction algorithm to
compute the buy-sell boundaries and value functions of the maximization problem.

The rest of the paper is organized as follows. Section 2 presents the utility based ap-
proach to pricing options under a geometric Lévy process with proportional transaction
costs and its corresponding singular stochastic control problems. Section 3 introduces
for the negative exponential utility function a coupled backward induction algorithm
to solve the control problems with much less computational cost. Section 4 provides
intensive simulation studies that investigate the impact of jump component in the stock
price process and transaction costs on option price and the implied hedging costs. Some
concluding remarks are given in Section 5.

2 A utility-based approach to option pricing with proportional trans-
action costs

2.1 Option pricing via utility maximization

Suppose that an investor is provided with an opportunity to enter into a position in a
European call option written on a stock with expiration date T and strike price K. The
price of the stock is assumed to follow a geometric Lévy process

\[ dS_t = \alpha S_t dt + \sigma S_t dW_t + S_t \int_{-1}^{\infty} \eta \tilde{N}(dt, d\eta). \]  \hspace{1cm} (1)

Here the mean rate of return \( \alpha > 0 \) and the volatility \( \sigma > 0 \) are constants, and \( W_t \) is
a standard Brownian motion on a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) with \( W_0 = 0 \).
The Poisson random measure $\tilde{N}$ is $\mathcal{F}_t$-centered, that is,

$$\tilde{N}(t, A) = N(t, A) - E[N(t, A)] = N(t, A) - tv(A),$$

in which Poisson random measure $N(t, A)$ measures the number of jumps with amplitude in $A \subset (-1, \infty)$ up to and including time $t$, $N$ has a time-homogeneous intensity, $E[N(t, A)] = tv(A)$, and $v$ is the Lévy measure associated to $N$. Note that to remain the price process $S_t > 0$ for all $t \geq 0$, we only allow jump sizes $\eta > -1$. We also assume that, for technical convenience,

$$\int_{-1}^{\infty} 1 \vee \eta^2 d\nu(\eta) < \infty.$$

In the presence of proportional transaction costs, the investor pays $0 < \zeta < 1$ and $0 < \mu < 1$ of the dollar value transacted on purchase and sale of the underlying stock, respectively. Let $x_t$ denote the number of shares held in stock and $y_t$ the dollar value of the investment in bond which pays a fixed risk-free rate $r$. The investor’s position $(x_t, y_t)$ in stock and bond is driven by:

$$dx_t = dL_t - dM_t,$$  \hspace{1cm} (2)

$$dy_t = ry_t dt - aS_t dL_t + bS_t dM_t,$$ \hspace{1cm} (3)

with $a = 1 + \zeta$ and $b = 1 - \mu$, where $L_t$ and $M_t$ are nondecreasing and non-anticipating processes and represent the cumulative numbers of shares of stock bought or sold, respectively, within the time interval $[0, t]$, $0 \leq t \leq T$. The process $(L_t, M_t)$ provides us an admissible trading strategy. For convenience, we denote $\mathcal{T}(y_0)$ the set of admissible trading strategies that an investor starts at time zero with amount $y_0$ of the investment in bond and zero holdings in stock. Then equations (1), (2) and (3) compose the market model in the time interval $[0, T]$, which describes a degenerate Lévy process in $\mathbb{R}^3$.

Denote the terminal settlement value of the stock and option by $Z^i(S_T, x_T)$, where $i = 0, s$ and $b$ indicates the investor’s position in the option: no call option, short call, and long call, respectively. When there is no option position, or $i = 0$, the liquidated value of stock is given by

$$Z^0(S, x) = xS(a\mathbb{1}_{x < 0} + b\mathbb{1}_{x \geq 0}).$$  \hspace{1cm} (4)

If the option is cash settled, the option writer delivers $(S_T - K)^+$ in cash at $T$, so

$$Z^i(S, x) = Z^0(S, x) - (S - K)\Delta^i(S), \hspace{1cm} i = s, b,$$  \hspace{1cm} (5)
where $\Delta^i(S) = \mathbb{I}_{\{S>K\}}$ (short call) and $\Delta^b(S) = -\mathbb{I}_{\{S>K\}}$ (long call). If the option is \textit{asset settled}, then the writer delivers one share of stock in return for a payment of $K$ when the buyer exercises the option at maturity $T$, so

$$Z^i(S,x) = Z^0(S, x - \Delta^i(S)) + K\Delta^i(S), \quad i = s, b.$$  

(6)

Note that the cases of (4)-(6) imply the trade of 0 share of stock for $i = 0$, $s$ and $b$ with cash settlement, and the trade of $\Delta^i(S_T)$ share of stock for $i = s$ and $b$ with asset settlement. Using the self-financing argument of Bensaid et al. (1992) and assuming that the investor can choose any time in $[t,T]$ to trade and the first trade after time $t$ occurs at time $\tau$, we then have $x_u = x_t$ for all $u \in [t,\tau)$, and the dollar value of the investment at time $\tau$ is given by

$$y_\tau = y_t e^{r(\tau-t)} - S_\tau(a d L_\tau - b d M_\tau),$$

in which $S_\tau(a d L_\tau - b d M_\tau)$ is the cost of the first trade. In general, suppose trades after time $\tau_0 := t$ occur at times $\tau_1 < \tau_2 < \cdots < \tau_n$, we have

$$y_T = y_t e^{r(T-t)} - \sum_{i=0}^n e^{r(T-\tau_i)} S_{\tau_i}(a d L_{\tau_i} - b d M_{\tau_i}).$$

Extending the argument to continuous time and continuous states yields

$$y_T = y_t e^{r(T-t)} - \Psi(L,M;t,T)$$

(7)

where

$$\Psi(L,M;t,T) = a \int_{[t,T]} e^{r(T-u)} S_u d L_u - b \int_{[t,T]} e^{r(T-u)} S_u d M_u$$

(8)

is the total trading cost incurred over $[t,T]$. In the mean while, the total hedging cost incurred in $[t,T]$ is expressed as

$$C^i(L,M;t,T) = \Psi(L,M;t,T) - Z^i(S_T,x_T), \quad i = 0, s, b.$$  

(9)

Therefore, combining (7)-(9) yields the investor’s terminal wealth in terms of the total hedging cost

$$y_T + Z^i(S_T,x_T) = y_t e^{r(T-t)} - C^i(L,M;t,T), \quad i = 0, s, b,$$

Suppose that the writer’s utility $U : \mathbb{R} \rightarrow \mathbb{R}$ is a concave and increasing function with $U(0) = 0$. We assume that the investor’s goal is to maximize the expected utility of terminal wealth under the market model (1)-(3)

$$V^i(t,S,x,y) = \sup_{(L_t,M_t) \in T(y_0)} \mathbb{E}[U(y_T + Z^i(S_T,x_T))|S_t = S, x_t = x, y_t = y],$$

(10)
which corresponds to no call option \((i = 0)\), short call \((i = s)\) and long call \((i = b)\). With the given utility function (10), the option price can be derived from the indifference argument which is similar to utility equivalence pricing principle in economics. In particular, the writing price of an option is defined as the amount of money that makes the investor indifferent, in terms of expected utility, between entering into the market with and without writing the option. At this reservation price, the investor is indifferent between selling (or buying) an option and doing nothing. Denote the reservation price of selling (or buying) an option as the amount of cash \(P_s\) (or \(P_b\)) required initially to provided the same expected utility as not enter into this position to the investor, \(P_s\) and \(P_b\) satisfy the following equations:

\[
V_s(0, S, x, y + P_s(S, x)) = V^0(0, S, x, y) = V_b(0, S, x, y - P_b(S, x)).
\]

We denote that the set of solutions of equations (1), (2) and (3) as

\[
T_K = \{(S_t, x_t, y_t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} | y_t + Z^i(x_t, S_t) < -K\},
\]

where \(K\) is a constant that may depend on the strategy \((L_t, M_t)\).

### 2.2 The Hamilton-Jacobi-Bellman equations and free boundary problems

We now derive the Hamilton-Jacobi-Bellman equations, associated with the stochastic control problems, for the utility maximization problem (10). Consider a class of trading strategies such that \(L_t\) and \(M_t\) are absolutely continuous processes, given by

\[
L_t = \int_0^t l_u du \quad \text{and} \quad M_t = \int_0^t m_u du,
\]

where \(l_u\) and \(m_u\) are positive and uniformly bounded by \(\xi < \infty\). Then equations (1)-(3) consist of a vector stochastic differential equation with controlled drift, and the Bellman equation for a value function \(V^i\) is given by

\[
\max_{l_t, m_t} \left\{ \left( P_1 V^i \right) l_t - \left( Q_1 V^i \right) m_t \right\} + O_1 V^i = 0,
\]

for \((t, S_t, x_t, y_t) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+,\) in which operators \(P_1, Q_1\) and \(O_1\) are defined as

\[
P_1 \phi := \frac{\partial \phi}{\partial x} - aS_t \frac{\partial \phi}{\partial y}, \quad Q_1 \phi := \frac{\partial \phi}{\partial x} - bS_t \frac{\partial \phi}{\partial y},
\]

\[
O_1 \phi := \frac{\partial \phi}{\partial t} + ry \frac{\partial \phi}{\partial y} + \alpha S \frac{\partial \phi}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \phi}{\partial S^2}
\]

\[
+ \int_{-1}^{\infty} \left[ \phi(t, S(1 + \eta), x, y) - \phi(t, S, x, y) - \eta S \frac{\partial \phi}{\partial S} \right] d\nu(\eta).
\]
respectively. Then the optimal trading strategy is determined by the following cases (Note that all other combinations of inequalities are impossible, since the value functions are increasing functions of \(x\) and \(y\)):

(i) buying stock at the maximum possible rate \(l_t = \xi\) and \(m_t = 0\) when \(\mathcal{P}_1 V \geq 0\) and \(\mathcal{Q}_1 V > 0\);

(ii) selling stock at the maximum possible rate \(m_t = \xi\) and \(l_t = 0\) when \(\mathcal{P}_1 V < 0\) and \(\mathcal{Q}_1 V \geq 0\);

(iii) doing nothing, that is \(m_t = l_t = 0\) when \(\mathcal{P}_1 V \leq 0\) and \(\mathcal{Q}_1 V \leq 0\);

The above argument shows that the optimization problem (10) is a free boundary problem. Besides, the state space \([0, T] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}\) is partitioned into buy, sell and no-transaction regions, which are characterized by inequalities in (i)-(iii), respectively. For sufficiently large \(\xi\), the state space remains divided into a buy region \(\mathcal{B}\), a sell region \(\mathcal{S}\), and a no-transaction region \(\mathcal{N}\), and the optimal trading strategy requires an immediate transaction to the boundary of the buy region, \(\partial \mathcal{B}\) or that of the sell region \(\partial \mathcal{S}\), if the state is in region \(\mathcal{B}\) or \(\mathcal{S}\). Hence the value function \(V^i(t, S, x, y)\) satisfy

\[
V^i(t, S, x, y) = V^i(t, S, x + \epsilon_x, y - aS\epsilon_x) \quad \text{in } \mathcal{B},
\]

\[
V^i(t, S, x, y) = V^i(t, S, x - \epsilon_x, y + bS\epsilon_x) \quad \text{in } \mathcal{S},
\]

in which \(\epsilon_x\) (the number of shares bought or sold by the investor) can take any positive value up to the number required to take the state to \(\partial \mathcal{B}\) or \(\partial \mathcal{S}\). Instantaneous transaction from the interior of the buy (or sell) region to the buy (or sell) boundary suggests equations \(\mathcal{P}_1 V^i = 0\) in \(\mathcal{B}\) and \(\mathcal{Q}_1 V^i = 0\) in \(\mathcal{S}\). In the no-transaction region \(\mathcal{N}\), the value function follows equation (12) for the trading strategies \(l_t = m_t = 0\), and hence becomes \(\mathcal{O}_1 V^i = 0\). Therefore, the above derived equations can be condensed into the following partial integro-differential equations (PIDE):

\[
\max \left\{ \mathcal{P}_1 V^i, -\mathcal{Q}_1 V^i, \mathcal{O}_1 V^i \right\} = 0 \quad (13)
\]

for \((t, S_t, x_t, y_t) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+\). The above discussion also yields the following free boundary problem for the singular stochastic control value function \(V^i\):

\[
\begin{cases}
\mathcal{O}_1 V^i = 0 & \text{in } \mathcal{N} \\
\mathcal{P}_1 V^i = 0 & \text{in } \mathcal{B} \\
\mathcal{Q}_1 V^i = 0 & \text{in } \mathcal{S} \\
V^i(T, S, x, y) = U(y + Z^i(S, x)).
\end{cases} \quad (14)
\]
Actually, we can further show that the value functions $V^i$ given by (10) are weak solutions of the variational inequality (13) in the notion of viscosity solutions.

**Theorem 1.** The value function $V^i$ is a constrained viscosity solution of

$$\max \{ P_1 \phi, -Q_1 \phi, O_1 \phi \} = 0 \quad (15)$$

on $[0, T] \times \mathcal{T}_K$.

### 2.3 Solutions for the negative exponential utility function

We further assume that the investor has the negative exponential utility function

$$U(z) = 1 - e^{-\gamma z}, \quad (16)$$

in which $\gamma$ is the constant absolute risk aversion (CARA) parameter. For equation (13), this utility function can reduce much of computational effort and is simple to interpret. Furthermore, the option price based on the exponential utility function is a good approximation to the price implied by any hyperbolic absolute risk aversion utility function with the same level of absolute risk aversion; see Andersen and Damgaard (1999).

Since this utility function will be used for algorithm design and convergence argument in Section 3 and also simulation studies in Section 4, we show below that the value function $V^i$ is also the unique bounded constrained viscosity solution of (15) for the exponential utility function (16).

**Theorem 2.** Let $u$ be a bounded upper semicontinuous viscosity subsolution of (15), and $v$ a bounded from below lower semicontinuous viscosity supersolution of (15), such that $u(T, S, x, y) \leq v(T, S, x, y)$ for all $(S, x, y) \in \mathcal{T}_K$. Then $u \leq v$ on $[0, T] \times \mathcal{T}_K$.

Note that for the utility function (16), the definition of the value function (10) can be expressed as

$$V^i(t, S, x, y) = 1 - \exp\{ -\gamma ye^{r(T-t)} \} H^i(t, S, x), \quad (17)$$

where

$$H^i(t, S, x) := \inf_{L,M} E\{ \exp[-\gamma(yT + Z^i(S_T,x_T) - ye^{r(T-t)})] S_t = S, x_t = x \}
= 1 - V^i(t, S, x, 0).$$

Plugging (17) into (13) and defining the following operators for $\psi(t, S, x)$ on $[0, T] \times \mathbb{R} \times \mathbb{R}$,

$$P_2 \psi := \frac{\partial \psi}{\partial x} + a\gamma Se^{r(T-t)} \psi, \quad Q_2 \psi := \frac{\partial \psi}{\partial x} + b\gamma Se^{r(T-t)} \psi,$$
\[
O_2\psi := \frac{\partial \psi}{\partial t} + aS \frac{\partial \psi}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \psi}{\partial S^2} + \int_{-1}^{\infty} \left[ \psi(t, S(1 + \eta), x) - \psi(t, S, x) - \eta S \frac{\partial \psi}{\partial S} \right] d\nu(\eta),
\]
we obtain an variational inequality
\[
\min \left\{ P_2^i H^i, -Q_2^i H^i, O_2^i H^i \right\} = 0, \quad (18)
\]
and the corresponding free boundary problem for \( H^i(t, S, x) \),
\[
\begin{aligned}
O_2^i H^i(t, S, x) &= 0 \quad &x \in [X_b(t, S), X_s(t, S)], \\
P_2^i H^i(t, S, x) &= 0 \quad &x \leq X_b(t, S), \\
Q_2^i H^i(t, S, x) &= 0 \quad &x \geq X_s(t, S), \\
H^i(T, S, x) &= \exp\{-\gamma Z^i(S, x)\},
\end{aligned} \quad (19)
\]
in which \( X_b(t, S) \) and \( X_s(t, S) \) are the buy and sell boundaries, respectively; see Davis et al. (1993). Moreover, combining (17) with (11) yields the reservation buying and selling prices
\[
\begin{aligned}
P^b(S, x) &= -\gamma^{-1} e^{-r T} \log \left[ \frac{H^b(0, S, x)}{H^0(0, S, x)} \right], \quad (20) \\
P^s(S, x) &= \gamma^{-1} e^{-r T} \log \left[ \frac{H^s(0, S, x)}{H^0(0, S, x)} \right]. \quad (21)
\end{aligned}
\]
In the limit as \( \gamma \to \infty \), Bouchard et al. (2001) have shown that the reservation price converges to the sum of the liquidation value of the initial endowment and the super-replication price of the option.

3 A numerical algorithm

Note that when the stock price process \( S_t \) follows a diffusion-only process, the free boundary problem (19) reduces to the one in Davis et al. (1993). To solve the option pricing problem under diffusions, Kushner and Dupuis (1992), Davis et al. (1993) and Zakamouline (2006) have developed a numerical scheme that involves a consistent Markov chain approximation of continuous-time price processes and then solved an appropriate optimization problem by discrete time dynamic programming. In particular, their scheme is based on weak convergence of probability measures and uses Markov chain approximation and discrete time dynamic programming algorithm. Though the extension of the scheme from diffusion-only to jump diffusion processes is possible, it is computationally expensive as the solutions of the singular stochastic control problem (19) require the determination of both when to apply control and how much control to apply.
In this section, we introduce a coupled backward induction algorithm that can substantially reduce computational complexities. The basic idea of coupled backward induction algorithm was introduced by Lai and Lim (2009). Specifically, based on a diffusion-only stock price process, Lai and Lim (2009) used utility maximization and cost constraint argument to derive the free boundary problem for singular stochastic control value functions, and then connected the free boundary problem to an optimal stopping problem. We employ the similar idea here and connect the free boundary problem (19) to an optimal stopping problem. However, since the underlying stock price process in our model involves both jumps and diffusions, the diffusion-based numerical scheme in Lai and Lim (2009) can not be applied to problem (19). To solve this issue, we choose the numerical scheme of Bernoulli walk with jumps in Aitsahlia and Runnemo (2007) for discretized jump-diffusion processes. In such way, we extend the idea of coupled backward induction algorithm to geometric Lévy processes.

3.1 Change of variables

To solve the free boundary problem (19), we apply the change of variables as follows:

\[
\begin{align*}
\rho &= \frac{r}{\sigma^2}, \quad s = \sigma^2(t - T), \quad z = \log(S/K) - (\alpha - \beta - \frac{1}{2})s, \\
\theta &= \frac{\alpha}{\sigma^2}, \quad \beta = \frac{1}{\sigma^2} \int_{-1}^{\infty} \eta d\nu(\eta),
\end{align*}
\] (22)

and let \( h^i(s, z, x) = H^i(t, S, x) \). Then inequality (18) becomes

\[
\min \left\{ P^3 h^i, -Q^3 h^i, O^3 h^i \right\} = 0,
\] (23)

and equation (19) can be rewritten as

\[
\begin{align*}
O^3 h^i(s, z, x) &= 0 & x \in [X_b(s, z), X_z(s, z)] \\
P^3 h^i(s, z, x) &= 0 & x \leq X_b(s, z) \\
Q^3 h^i(s, z, x) &= 0 & x \geq X_z(s, z) \\
h^i(0, z, x) &= \exp\{-\gamma KA^i(z, x)\},
\end{align*}
\] (24)

where the operator \( P^3, Q^3 \) and \( O^3 \) are defined as follows:

\[
P^3 \psi := \frac{\partial \psi}{\partial x}(s, z, x) + a\gamma K e^{\rho s + (\theta - \rho - \beta - 0.5)s} \psi(s, z, x),
\] (25)

\[
Q^3 \psi := \frac{\partial \psi}{\partial x}(s, z, x) + b\gamma K e^{\rho s + (\theta - \rho - \beta - 0.5)s} \psi(s, z, x),
\] (26)
\[ O_3 \psi := \frac{\partial \psi}{\partial s} + \frac{1}{2} \frac{\partial^2 \psi}{\partial z^2} + \frac{1}{\sigma^2} \int_{-1}^{\infty} \left[ \psi(s, z + \log(1 + \eta), x) - \psi(s, z, x) \right] d\nu(\eta). \]  

(27)

Then, corresponding to definitions (4)-(6) of terminal wealth,

\[
\begin{align*}
A^0(z, x) &= x e^z (a I_{\{x < 0\}} + b I_{\{x \geq 0\}}), \\
A^i(z, x) &= A^0(z, x - D^i(z)) + D^i(z), \quad \text{for asset settlement \ } i = s, b, \\
A^i(z, x) &= A^0(z, x) - (e^z - 1) D^i(z), \quad \text{for cash settlement \ } i = s, b,
\end{align*}
\]

with \( D^s(z) = I_{\{z > 0\}} \) and \( D^b(z) = -I_{\{z > 0\}} \). Note that the operator \( O_3 \) can be viewed as the infinitesimal generator of a jump diffusion process defined through the stochastic differential equation,

\[ dz_s = d\tilde{w}_s + d\tilde{n}_s, \]

where \( \tilde{w}_s \) is a standard Brownian motion and \( \tilde{n}_s \) is a jump process with jump size scaled by \( 1/\sigma^2 \). This implies that while \((s, z)\) is inside the no-transaction region, the dynamics of \( h^i(s, z, x) \) is driven by the jump diffusion process \( \{z_s, s \leq 0\} \). Furthermore, it follows from (24) that, in the buy and sell regions,

\[
\begin{align*}
h^i(s, z, x) &= \exp\left\{ -a \gamma K e^{z + (\theta - \rho - \beta - 0.5)s} [x - X_b(s, z)] \right\} h^i(s, z, X_b(s, z)), \quad x \leq X_b(s, z), \\
h^i(s, z, x) &= \exp\left\{ -b \gamma K e^{z + (\theta - \rho - \beta - 0.5)s} [x - X_s(s, z)] \right\} h^i(s, z, X_s(s, z)), \quad x \geq X_s(s, z).
\end{align*}
\]

(28)

We next discuss how to get a solution for the free boundary problem (24). As mentioned in the beginning of this section, one way to solve (24) directly is to extend the numerical scheme that Kushner and Dupuis (1992), Davis et al. (1993) and Zakamouline (2006) developed for a diffusion-only process, which involves a consistent Markov chain approximation of continuous-time price processes and then solves an approximate optimization problem by discrete time dynamic programming. Since our setting here involves both jump and diffusion processes, it is too complicated to extend the numerical scheme in Kushner and Dupuis (1992), Davis et al. (1993) and Zakamouline (2006). Instead of solving (24) directly, we consider solving two partial integro-differential equations that are much easier than the original problem (24). The first partial integro-differential equation has the same form as (24) but the boundaries are assumed to be known. The second partial integro-differential equation is a free boundary problem with differential equations in the buy or sell regions are completely specified. Specifically, the second partial integro-differential equation is constructed as follows. Let \( w^i = \partial h^i / \partial x, \)
then $w^i$ satisfies the free boundary problem

$$
\begin{align*}
\mathcal{O}_3 w^i(s, z, x) &= 0 & x \in [X_b(s, z), X_s(s, z)] \\
w^i(s, z, x) &= -a \gamma K e^{x+(\theta-\rho-\beta-0.5)s} h^i(s, z, x) & x \leq X_b(s, z) \\
w^i(s, z, x) &= -b \gamma K e^{x+(\theta-\rho-\beta-0.5)s} h^i(s, z, x) & x \geq X_s(s, z) \\
w^i(0, z, x) &= -\gamma KB^i(z, x) h^i(0, z, x),
\end{align*}
$$

(29)

where $\mathcal{O}_3$ is defined by (25) and $B^i(z, x) = \partial A^i(z, x)/\partial x$ is given by

$$
\begin{align*}
B^0(z, x) &= A^0(z, x)/x, \\
B^i(z, x) &= B^0(z, x - D^i(z)), \text{ for asset settlement } i = s, b, \\
B^i(z, x) &= B^0(z, x), \text{ for cash settlement } i = s, b.
\end{align*}
$$

In the sequel, we fix $i = 0, s, b$ and drop the superscript $i$ in $w^i$ and $h^i$ for notational simplicity. The purpose of introducing problem (29) is that, when the function $h^i(s, z, x)$ is known, (29) becomes an optimal stopping problem associated with a Dynkin game (see Lai and Lim, 2009, Section 3), which can be easily solved by backward induction and random walk approximation (see Chernoff and Petkau, 1986; Ait-Sahalia and Lai, 1999; Lai et al., 2007).

### 3.2 A coupled backward induction algorithm

We now introduce a coupled backward induction algorithm that solves (24) and (29) to obtain the buy and sell boundaries $X_b(s, z)$ and $X_s(s, z)$ and the value function $h(s, z, x)$ simultaneously. Since the free boundary problem (29) with known $h(s, z, x)$ can be solved via random walk approximations, we consider a numerical scheme of Bernoulli walk with jumps recently proposed by Aitsahlia and Runnemo (2007).

Given a small $\delta > 0$, we discretize time and space as follows. Let $s_0 = 0$ and $s_j = s_{j-1} - \delta$ for $j \geq 1$ and set

$$
S_\delta = \{s_n, s_n - \delta, \ldots, \delta, 0\},
$$

$$
Z_\delta = \{\sqrt{\delta}j : j \text{ is an integer} \} = \{0, \pm \sqrt{\delta}, \pm 2\sqrt{\delta}, \ldots\}.
$$

Note that the possibility of jump in an interval of length $\delta$ can be approximated by $\beta \delta$ and hence the possibility of no jump in the interval is $1 - \beta \delta$. Let $S^\delta_j$ be the position of the approximating discrete process after the $n$th transition. On the grid $Z_\delta$, a jump occurs at the $(j + 1)$ transition if $S^\delta_j - S^\delta_{j+1} = \pm l\sqrt{\delta}$ for $l \geq 2$. We then approximate the jump distribution by partitioning the entire real line into intervals of length $\sqrt{\delta}$, whose
probability measure is assigned as the probability mass on the midpoints of the intervals. In particular, let $F$ be the cumulative probability distribution function of the jump size $Y$, we set

$$dF(l) = \begin{cases} 
F[(l + \frac{1}{2})\sqrt{\delta}] - F[(l - \frac{1}{2})\sqrt{\delta}], & |l| \geq 2, \\
F[(l + \frac{1}{2})\sqrt{\delta}] - F[(-l - \frac{1}{2})\sqrt{\delta}], & l = 0.
\end{cases}$$

(30)

We then assign the following probabilities

$$P\{S_{j+1} - S_j = \pm l\sqrt{\delta}\} = \begin{cases} 
\frac{1}{2}(1 - \beta\delta) & \text{if } l = 1, \\
\beta\delta dF(l) & \text{if } l \neq 1.
\end{cases}$$

In practice, we need to truncate the set of $l$ to a finite subset $\{l_{\min}, l_{\min} + 1, \ldots, l_{\max}\}$. In such case, we set $dF(l_{\min})$ and $dF(l_{\max})$ as

$$dF(l_{\min}) = P\{Y \leq (l_{\min} + 0.5)\sqrt{\delta}\},$$

$$dF(l_{\max}) = P\{Y > (l_{\max} - 0.5)\sqrt{\delta}\}.$$

We next use the above numerical scheme and introduce a backward induction to solve (29), provided $h(s, z, x)$ is known. Specifically, let $T_{\max}$ denote the largest expiration date of interest and take $\delta$ and $\epsilon > 0$ such that $N := \sigma^2 T_{\max}/\delta$ is an integer. The problem (29) with known $h(s, z, x)$ can be solved by the following backward induction; see Lai and Lim (2009). Let $X_\epsilon = \{0, \pm \epsilon, \pm 2\epsilon, \ldots \}$. For $i = 1, 2, \ldots, N$,

$$w(s_i, z, x) = \begin{cases} 
w_b(s_i, z, x) & \text{if } \bar{w}(s_i, z, x) < w_b(s_i, z, x), \\
w_s(s_i, z, x) & \text{if } \bar{w}(s_i, z, x) > w_s(s_i, z, x), \\
\bar{w}(s_i, z, x) & \text{otherwise,}
\end{cases}$$

where $x \in X_\epsilon$ and

$$w_b(s, z, x) = -a_\gamma Ke^{z+(\theta-\rho-\beta-1/2)}h(s, z, x),$$

$$w_s(s, z, x) = -b_\gamma Ke^{z+(\theta-\rho-\beta-1/2)}h(s, z, x),$$

$$\bar{w}(s_i, z_j, x_m) = \frac{1}{2}(1 - \beta\delta)\left[w(s_{i-1}, z_{j+1}, x_m) + w(s_{i-1}, z_{j-1}, x_m)\right] + \beta\delta \sum_{l=l_{\min}, l \neq \pm 1}^{l_{\max}} dF(l) \cdot w(s_{i-1}, z_{j+l}, x_m).$$

(33)

The boundaries in (29) is determined as follows: $X_b(s_i, z)$ is the largest $x$ for which $\bar{w}(s_i, z, x) \leq w_b(s_i, z, x)$ and $X_s(s_i, z)$ is the smallest $x$ for which $\bar{w}(s_i, z, x) \geq w_s(s_i, z, x)$.

We then solve numerically the problem (24), provided the boundaries $X_b(s, z)$ and $X_s(s, z)$ are given. In such case, the value function in (24) with provided boundaries
can also be solved by backward induction. In particular, for \( z \in Z_\delta \), define \( h(s_i, z, x) \) by (28) (with \( s \) replaced by \( s_i \)) if \( x \in X_e \) is outside the interval \([X_b(s_i, z), X_s(s_i, z)]\), and let \( h(s_i, z, x) = \tilde{h}(s_i, z, x) \) with

\[
\tilde{h}(s_i, z_j, x_q) = \frac{1}{2} (1 - \beta \delta) \left[ h(s_{i-1}, z_{j+1}, x_q) + h(s_{i-1}, z_{j-1}, x_q) \right] + \beta \delta \sum_{l=\min l \neq \pm 1}^{l_{\max}} dF(l) \cdot h(s_{i-1}, z_{j+l}, x_q),
\]

(34)

if \( x \in X_e \cap [X_b(s_i, z), X_s(s_i, z)] \). Defining \( \tilde{w}_b \) and \( \tilde{w}_s \) as in (31) and (32) but with \( h \) replaced by \( \tilde{h} \), it suggests the coupled backward induction algorithm described below to solve for \( X_b(s_i, z) \) and \( X_s(s_i, z) \), as well as to compute values of \( h(s_i, z, x) \) for \( x \in X_e \cap [X_b(s_i, z), X_s(s_i, z)] \).

**Algorithm.** For \( i = 1, 2, \ldots, N \) and \( z \in Z_\delta \):

(i) Starting at \( x_0 \in X_e \) with \( \tilde{w}(s_i, z, x_0) \), search for the first \( m \in \{1, 2, \ldots, \} \) (denoted by \( m^* \) for which \( \tilde{w}(s_i, z, x_0 + m^*\epsilon) \geq \tilde{w}_b(s_i, z, x_0 + m\epsilon) \)) and set \( X_b(s_i, z) = x_i + m^*\epsilon \).

(ii) Let \( x_m = X_b(s_i, z) + m\epsilon \) for \( m \in \{1, 2, \ldots, \} \). Compute, and store for use at \( s_{i+1} \), \( w(s_i, z, x_m) = \tilde{w}(s_i, z, x_m) \) as defined by (33) and \( h(s_i, z, x_m) = \tilde{h}(s_i, z, x_m) \) by (34). Search for the first \( m \) (denoted by \( m^* \)) for which \( \tilde{w}(s_i, z, x_m) \) and set \( X_s(s_i, z) = X_b(s_i, z) + m^*\epsilon \).

(iii) For \( x \in X_e \) outside the interval \([X_b(s_i, z), X_s(s_i, z)]\), compute \( h(s_i, z, x) \) by (28) and set \( w(s_i, z, x) = w_b(s_i, z, x) \) or \( w_s(s_i, z, x) \) as defined by (31) and (32) according to whether \( x \leq X_b(s_i, z) \) or \( x \geq X_s(s_i, z) \).

Note that, comparing to the discrete-time dynamic programming algorithms of Davis et al. (1993), Clewlow and Hodges (1997), and Zakamouline (2006) that need to perform an additional nonlinear optimization to identify the optimal trade size at each time step, the above algorithm avoids such optimization by solving the coupled problem for \((w, h)\) instead of just for \( h \) and hence is much easier to implement. The convergence of the algorithm can be shown by the following argument.

**Theorem 3.** The solution of the algorithm converges locally uniformly as \( \delta \to 0 \) to the unique continuous constrained viscosity solution of (24).
4 Numerical Study

We now perform a numerical study using the proposed algorithm. For illustration purpose, the price process of the underlying is assumed to follow a double exponential jump-diffusion process (Kou (2002)) with mean rate of return $\alpha > 0$ and volatility $\sigma > 0$

$$dS_t = \alpha S_t \, dt + \sigma S_t \, dW_t + S_t \, d\left(\sum_{i=1}^{N_t} (Q_i - 1)\right),$$

where $\alpha$ and $\sigma$ are the expected return and diffusion volatility of the underlying asset, $\{W_t; t \geq 0\}$ is a standard Brownian motion with $W_0 = 0$, $\{N_t; t \geq 0\}$ is a Poisson process with rate $\lambda$, $Q_t$ is a sequence of independent and identically distributed positive random variables such that the jump $Y = \log Q$ has asymmetric double exponential distribution with density

$$f_Y(y) = \begin{cases} 
  p \eta_1 e^{-\eta_1 y} & \text{if } y \geq 0, \\
  (1-p) \eta_2 e^{\eta_2 y} & \text{if } y < 0.
\end{cases}$$

Here $0 \leq p \leq 1$, and $p$ and $1 - p$ represent the probability of positive and negative jumps, respectively. The parameters $\eta_1$ and $\eta_2$ are assumed to satisfy $\eta_1 > 1$ and $\eta_2 > 0$. Note that under this specification, (30) becomes

$$dF(l) = \begin{cases} 
  (1-p)(e^{\eta_2(l+0.5)\sqrt{\delta}} - e^{\eta_2(l-0.5)\sqrt{\delta}}) & \text{if } l \leq -2, \\
  p(e^{-\eta_1(l-0.5)\sqrt{\delta}} - e^{-\eta_1(l+0.5)\sqrt{\delta}}) & \text{if } l \geq 2, \\
  (1-p)e^{1.5\eta_2 \sqrt{\delta}} - pe^{-1.5\eta_1 \sqrt{\delta}} & \text{if } l = 0,
\end{cases}$$

and the boundaries of the jump distribution is set as follows:

$$dF(l_{\min}) = (1-p) e^{\eta_2(l_{\min}+0.5)\sqrt{\delta}},$$
$$dF(l_{\max}) = pe^{-\eta_1(l_{\max}-0.5)\sqrt{\delta}}.$$

We then study the reservation price of a short call option with $K = 100$ and $T = 0.5$ under the process (35). To illustrate the algorithm, we choose $\alpha = r$, which let us characterize the optimal hedging strategy for the utility-maximization approach by a pair (rather than two pairs) of buy-sell boundaries. We further assume that $\alpha = r = 0$ and $\sigma = 0.3$ for the diffusion part of (35), which are same as the parameter configuration used by Clewlow and Hodges (1997). For the discretization of time, space and number of shares of stock, we use $\delta = \epsilon = 10^{-4}$ in the following study.
4.1 Impact of jumps

We first investigate the impact of jump components on the price of the call option and buy and sell boundaries. In this experiment, we assume the CARA parameter $\gamma = 1$ and transaction cost $\zeta = \mu = 0.01$. To discuss the impact of different rates and directions of jumps, we let $\eta_1 = \eta_2 = 25$ and consider different values of Poisson rate $\lambda$ and the probability $p$ of positive jumps.

Figure 1 shows the optimal buy (lower) and sell (upper) boundaries $X_b(t, S)$ and $X_s(t, S)$ for the short call with $p = 0.5$, $S_0 = 90, 100, 110$, and $\lambda$ with asset and cash settlements, respectively. For each pair of boundaries, the buy-region is below the buy boundary and the sell-region is above the sell boundary. The region between the buy and sell boundaries is the no-transaction region. In the asset settlement, when the call option is at the money, (i.e., $K = S = 100$), the buy and sell boundaries over time are symmetric around certain level and stretch out in the diffusion-only case (i.e., $\lambda = 0$); with the increase of $\lambda$, the no transaction region before expiration (i.e. $T - t = 0$) become narrower and boundaries began to stretch towards the inside of the no-transaction region. When the call option is in the money (i.e., $K = 100, S = 110$), buy and sell boundaries move upward to level 1 and two boundaries become 1 when the time to maturity ranges between 0.02 and 0; furthermore, with the increase of $\lambda$ from 0 to 10, the period during which buy and sell boundaries coincides become much shorter. Similar pattern is also found when the call option is out of the money (i.e., $K = 100, S = 90$), except that buy and sell boundaries in such case move downward to level 0. In the cash settlement, with the increase of $\lambda$, the no transaction region in the at-the-money case become much narrower when the option will expire very soon. Around the expiration, the no-transaction region of the in-the-money call option becomes wider, while that of the out-of-the-money call option shrink to 0. This fact shows that large price movement narrows down the no-transaction region, and indicates that liquidating any excess position as soon as possible is optimal for the option writer.

Figure 2 shows the optimal buy (lower) and sell (upper) boundaries $X_b(t, S)$ and $X_s(t, S)$ for the short call with $S_0 = 90, 100, 110$, and $p = 0.1, 0.3, 0.5, 0.9$ with asset and cash settlements, respectively. Note that with the increase of $p$, the trend of stock price movement changes from decreasing to increasing, the no-transaction regions around $T - t = 0.5$ move downward, whereas those around the expiration period move upward in both asset and cash settlements. The configurations of no-transaction regions for the in-, at-, and out-of-the-money options are very different. In the asset settlement, when the call option is at-the-money, the buy and sell boundaries have downward tails, but
Figure 1: Optimal buy (lower) and sell (upper) boundaries $X_b(t,S)$ and $X_s(t,S)$ from CARA utility function for a short call with asset (top four panels) and cash (bottom four panels) settlements, respectively, where $\gamma = 4, \zeta = \mu = 0.01, K = 100, S = 90$ (dashed line), $100$ (solid line), $110$ (dot-dash line), and $p = 0.5$. The Poisson rates in the panels (from top to bottom and left to right) are $\lambda = 0, 1, 5, 10, 0, 1, 5, 10$, respectively.
Figure 2: Optimal buy (lower) and sell (upper) boundaries $X_b(t,S)$ and $X_s(t,S)$ from CARA utility function for a short call with asset (top four panels) and cash (bottom four panels) settlements, respectively, where $\gamma = 4, \zeta = \mu = 0.01, K = 100, S = 90$ (dashed line), 100 (solid line), 110 (dot-dash line), and $\lambda = 4$. The rates of jumps (from top to bottom and left to right) are $p = 0.1, 0.3, 0.5, 0.9, 0.1, 0.3, 0.5, 0.9$, respectively.
these tails move up with the increase of $p$. When the option is in-the-money (out-of-the-money), it has similar pattern and more tails of the sell (buy) boundary become flat when $p$ increases.

Besides the buy and sell boundaries, we also consider the impact of jumps on the prices of the short call options. Table 1 shows the prices of the short call option in Figures 1 and 2 with $S_0 = 100$. We notice three facts from the table. First, when other conditions are same, the call option price with cash settlement is more expensive than that with asset settlement, this is because the strategy in the cash settlement involves higher hedging cost and hence yields higher option price. Second, more frequent jumps in price (i.e., larger $\lambda$) leads to higher option price, as more frequent jumps in price narrow down the no-transaction region and imply higher hedging cost and higher option price. Third, higher probability of positive jumps leads to lower option price, as more frequent positive jumps decreases the number of $S_t$ hitting buying boundaries and hence reduce the hedging cost and option price. To have a better intuition on the impact of jump intensity $\lambda$ and positive probability $p$ on option price, we show in Figure 3 the reservation price for the short call option with $\zeta = \mu = 0.01, S = K = 100, T = 0.5$ under asset settlement. We can see that when the probability of positive jumps $p$ is less than 0.5 (i.e., stock prices have the tendency of going down), the reservation price increases with the increase of $\lambda$; while if $p > 0.5$ (i.e., stock prices have the tendency of moving up), the reservation price decreases with the increase of $\lambda$. 

<table>
<thead>
<tr>
<th></th>
<th>$p = 0.5$</th>
<th>$\lambda = 0$</th>
<th>$\lambda = 1$</th>
<th>$\lambda = 5$</th>
<th>$\lambda = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price for Asset Settlement option</td>
<td>5.98</td>
<td>6.13</td>
<td>6.58</td>
<td>7.05</td>
<td></td>
</tr>
<tr>
<td>Price for Cash Settlement option</td>
<td>6.31</td>
<td>6.46</td>
<td>6.92</td>
<td>7.39</td>
<td></td>
</tr>
<tr>
<td>$\lambda = 4$</td>
<td>$p = 0.1$</td>
<td>$p = 0.3$</td>
<td>$p = 0.5$</td>
<td>$p = 0.9$</td>
<td></td>
</tr>
<tr>
<td>Price for Asset Settlement option</td>
<td>9.16</td>
<td>7.75</td>
<td>6.48</td>
<td>4.33</td>
<td></td>
</tr>
<tr>
<td>Price for Cash Settlement option</td>
<td>9.59</td>
<td>8.13</td>
<td>6.82</td>
<td>4.59</td>
<td></td>
</tr>
</tbody>
</table>
Figure 3: Reservation price for short call option with $\zeta = \mu = 0.01$, $S = K = 100$, $T = 0.5$ and asset settlement. The probability of positive $p = 0, \ldots, l$ and the rate of jumps $\lambda = 0, \ldots, 10$.

4.2 Impact of the transaction cost

We further study the impact of transaction cost on the buy and sell boundaries by considering the following parameter configuration:

$$K = 100, \quad S = 100, \quad \gamma = 1, \quad \lambda = 5, \quad p = 0.5, \quad \eta_1 = 25, \quad \eta_2 = 25$$

in asset and cash settlements, and compute the optimal buy and sell boundaries with the transaction cost $\zeta = \mu = 0, 0.01, \ldots, 0.1$. Figure 4 shows the optimal buy and sell boundaries for a short call option with asset and cash settlements. We find that the impact of transaction cost is always monotone for the sell boundary in both asset and cash settlements, while such impact for the buy boundary is monotone in cash settlement but not in asset settlement. This results in larger no-transaction regions for larger transaction costs, which is consistent with the fact that larger transaction costs will reduce the enthusiasm of trading. Furthermore, when transaction costs converge to 0, the buy and sell boundaries converge to the curve of hedging delta.
Figure 4: Optimal buy (left panels) and sell (right panels) boundaries for a short call with asset (upper panels) and cash (lower panels) settlement. The transaction cost is $\zeta = \mu = 0, 0.01, \ldots, 0.1$.

4.3 Impact of the risk aversion parameter

We now study the impact of utility functions with different CARA parameters on the buy and sell boundaries. Figure 5 shows the optimal buy (lower) and sell (upper) boundaries $X_b(t, S)$ and $X_s(t, S)$ for the short call with $S_0 = 90, 100, 110$, and $\gamma = 0.5, 1, 2, 4$ with asset and cash settlements, respectively. An interesting phenomenon here is that it seems that CARA parameters seem have little impact on buy and sell boundaries in the beginning of the trading process (i.e. $T - t = 0.5$) in both settlement, but the impact become significant when the option is near expiration ($T - t < 0.05$). In particular, we notice that the no-transaction regions for at-the-money options become much narrower when $\gamma$ gets larger, while the no-transaction regions for in- and out-of-the-money options do not change as significantly as for at-the-money options. This indicates that when the agent becomes more risk-averse, he tends to buy or sell stocks before the expiration so that the loss caused by changes of the option value can be avoided or mitigated.
Figure 5: Optimal buy (lower) and sell (upper) boundaries $X_b(t,S)$ and $X_s(t,S)$ from CARA utility function for a short call with asset (top four panels) and cash (bottom four panels) settlements, respectively, where $\zeta = \mu = 0.01$, $K = 100$, $S = 90$ (dashed line), 100 (solid line), 110 (dot-dash line), $\lambda = 4$, and $p = 0.5$. The CARA parameters in the panels (from top to bottom and left to right) are $\gamma = 0.5, 1, 2, 4, 0.5, 1, 2, 4$, respectively.
4.4 Total hedging cost

We now study the total hedging cost of the optimal trading strategies based on the buy and sell boundaries in Section 2. In particular, we consider a short call option with strike \( K = 100 \) and time to maturity \( T = 0.5 \). We also assume that the true stock price process is given by the double exponential jump diffusion processes (35) with parameters \( S_0 = 100, \eta_1 = \eta_2 = 25, \) and \( \lambda = 1, 3, 5, 10, 12, \) respectively. The transaction costs are specified by \( \zeta = \mu = 0.01 \). We then consider trading boundaries computed from a “correctly-” and an “incorrectly-” specified pricing models. The “correctly” specified model uses the jump diffusion process (35) with given parameters and is solved by the procedures in Sections 2 and 3; the “incorrectly” specified model assumes no jump in the price process or the price process follow a geometric Brownian motion, \( dS_t = \alpha S_t dt + \sigma S_t dW_t, \) and is solved by the procedures in Davis et al. (1993). Then for each trading boundaries generated from both models, we simulate \( 10^5 \) price paths according to the double exponential jump diffusion processes (35), and compute the total hedging cost \( \hat{C} \) according to (9) and the total number of trades \( \hat{\kappa} \). We summarize the statistic on \( \hat{C} \) and \( \hat{\kappa} \) from the “correctly-” and the “incorrectly-” specified models in the second and the third columns of Table 2, respectively. Note that the correctly specified models always give larger total number of trades but lower total hedging cost, comparing with those of the incorrectly specified models.

5 Concluding Remarks

We consider the problem of European option pricing in the presence of proportional transaction cost when the price of the underlying follows a jump diffusion process. Using an approach that is based on maximization of the expected utility of terminal wealth, we transform the option pricing into stochastic optimal control problems, and argue that the value functions of these problems are the solutions of a free boundary problem which consists of a partial integro-differential equation and different boundary conditions. We develop a coupled backward induction algorithm to solve the singular stochastic control problems associated with utility maximization, and compute the value function and no-transaction boundaries. The developed algorithm computes the value functions and the trading boundaries of the stochastic control problems simultaneously and hence greatly reduces the computational cost.
Table 2: Summary statistics of simulation on total hedging cost.

<table>
<thead>
<tr>
<th>λ</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>10</th>
<th>12</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(0.004)</td>
<td>(0.005)</td>
<td>(0.005)</td>
<td>(0.007)</td>
<td>(0.007)</td>
<td>(0.004)</td>
<td>(0.005)</td>
<td>(0.005)</td>
<td>(0.006)</td>
<td>(0.007)</td>
</tr>
<tr>
<td>̅κ</td>
<td>88.24</td>
<td>89.81</td>
<td>91.12</td>
<td>93.70</td>
<td>94.75</td>
<td>85.90</td>
<td>85.95</td>
<td>86.13</td>
<td>86.52</td>
<td>86.57</td>
</tr>
<tr>
<td></td>
<td>(0.04)</td>
<td>(0.04)</td>
<td>(0.04)</td>
<td>(0.05)</td>
<td>(0.05)</td>
<td>(0.04)</td>
<td>(0.04)</td>
<td>(0.04)</td>
<td>(0.05)</td>
<td>(0.05)</td>
</tr>
</tbody>
</table>

(a) Asset settlement

|     | (0.004)| (0.005)| (0.006)| (0.007)| (0.007)| (0.004)| (0.005)| (0.005)| (0.006)| (0.007)|
| ̅κ  | 86.13 | 87.72 | 89.16 | 91.85 | 92.83 | 84.00 | 84.14 | 84.22 | 84.51 | 84.66 |
|     | (0.05) | (0.05) | (0.05) | (0.05) | (0.05) | (0.04) | (0.05) | (0.05) | (0.05) | (0.05) |

(b) Cash settlement

---

Declarations of Interest

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

Acknowledgement

The authors thank for an associate editor and two anonymous referees for very helpful comment. This research is supported by the National Science Foundation under grants DMS-0906593 and DMS-1206321 at SUNY at Stony Brook.

References


25


Appendix of "European Option Pricing under Geometric Lévy Processes with Proportional Transaction Costs"

Proofs of Theorems 1-3

Sketched Proof of theorem 1. In our discussion, the state is $W = (t, w)$, where $w = (S, x, y)$. Let $W_0 = (t_0, S_0, x_0, y_0) \in [0, T] \times \mathcal{T}_K$.

We first show that $V^1$ is a viscosity subsolution of (15) on $[0, T] \times \mathcal{T}_K$, or, for any smooth function $\phi(W)$, such that $V^1(W) - \phi(W)$ has a local maximum at $W_0 \in [0, T] \times \mathcal{T}_K$.

$$\max\{\mathcal{P}_1\phi(W_0), -\mathcal{Q}_1\phi(W_0), \mathcal{O}_1\phi(W_0)\} \geq 0.$$ (36)

Suppose that (36) is wrong, then for smooth function $\phi$ such that $V^1(W_0) = \phi(W_0)$ and $V^1(W) \leq \phi(W)$ on $[0, T] \times \mathcal{T}_K$, if $\mathcal{P}_1\phi(W_0) < 0$, $\mathcal{Q}_1\phi(W_0) > 0$, there must exist $\delta > 0$, such that $\mathcal{O}_1\phi(W_0) < -\delta$. By smoothness of $\phi$, there exists a neighborhood $\mathcal{R}(W_0)$ of $W_0$, such that for $W \in \mathcal{R}(W_0)$, we have $\mathcal{P}_1\phi(W) < 0$, $\mathcal{Q}_1\phi(W) > 0$, and $\mathcal{O}_1\phi(W) < -\delta$.

Note that by our construction, there exists an optimal trading strategy corresponding to the processes $L^*(t), M^*(t)$, such that $W^*_0(t) = (t, B^*(t), x^*(t), y^*(t))$ is the optimal trajectory with $W^*_0(t_0) = W_0$. We show that $W^*_0(t)$ has no jumps, almost surely, at $W_0 = W^*_0(t_0)$. Denote the event that the optimal trajectory $W^*_0(t)$ has a jump of size $\epsilon$ along the direction $(0, 0, 1, -aS_0)$ by $I(\omega)$, then since $V^1(W) \leq \phi(W)$ for all $W \in \mathcal{E}(W_0)$ and $V^1(W_0) = \phi(W_0)$, by the principle of dynamic programming, $\int_{I(\omega)} \phi(t_0, S_0, x_0 + \epsilon, y_0 - aS_0\epsilon)dP \geq \int_{I(\omega)} \phi(t_0, S_0, x_0, y_0)dP$. Therefore, letting $\epsilon \to 0$ and using Fatou’s lemma, we have $(\mathcal{P}_1\phi(W_0))P(I) \geq 0$, and therefore $P(I) = 0$.

Now, define $\tau(\omega) := \inf\{t \in (t_0, T)|W^*_0(t) \notin \mathcal{E}(W_0)\}$, $\tau(\omega)$ is positive almost surely. Therefore,

$$-\delta E(\tau) > \mathbb{E} \int_{t_0}^{T} \mathcal{P}_1(W^*_0(t))dL^*(t) - \mathbb{E} \int_{t_0}^{T} \mathcal{Q}_1(W^*_0(t))dM^*(t) + \mathbb{E} \int_{t_0}^{T} \mathcal{O}_1(W^*_0(t))dt$$

$$= \mathbb{E}A_1 - \mathbb{E}A_2 + \mathbb{E}A_3.$$

Applying Itô’s formula to $\phi(W)$, we have

$$E(\phi(W^*_0(\tau))) \leq V^1(W_0) + \mathbb{E}A_1 - \mathbb{E}A_2 + \mathbb{E}A_3 < V^1(W_0) - \delta E(\tau),$$

which contradicts with the optimality of $L^*(t)$ and $M^*(t)$ and the dynamic programming principle. Therefore, at least one of items inside the maximum operator of (36) is nonnegative. Hence the value function $V^1$ is a viscosity subsolution of (15).
To show that $V^1$ is a viscosity supersolution of (15), we need to show that, for all smooth functions $\phi(W)$ such that $V^1(W) - \phi(W)$ has a local minimum at $W_0 \in [0, T] \times \mathcal{T}_K$,

$$\max\{\mathcal{P}_1\phi(W_0), -\mathcal{Q}_1\phi(W_0), \mathcal{O}_1\phi(W_0)\} \leq 0. \quad (37)$$

Assume $V^1(W_0) = \phi(W_0)$ and $V^1 \geq \phi$ on $[0, T] \times \mathcal{T}_K$. Consider the trading strategy $L(t) = L_0 > 0$, $t_0 \leq t \leq T$, and $M(t) = M_0 > 0$, $t_0 \leq t \leq T$. By the dynamic programming principle,

$$V^1(t_0, S_0, x_0, y_0) \geq V^1(t_0, S_0, x_0 + L_0, y_0 - aS_0L_0).$$

This inequality also holds for $\phi(t, S, x, y)$. Letting $L_0 \to 0$, we get $\mathcal{P}_1\phi(W_0) \leq 0$. Similarly, considering the strategy $L(t) = 0$, $t_0 \leq t \leq T$, and $M(t) = M_0 > 0$, $t_0 \leq t \leq T$, we can argue that $-\mathcal{Q}_1\phi(W_0) \leq 0$. Then for the case where no trading occurs. By the dynamic programming principle $\mathbb{E}(V^1(W^d_0(t))) \leq V^1(t_0, S_0, x_0, y_0)$, where $W^d_0(t)$ is the state trajectory when $L(t) = M(t) = 0$, $t_0 \leq t \leq T$ and $W^d_0(t) \in \mathcal{E}(W_0)$. Applying Itô’s formula on $\phi(t, S, x, y)$, we get $\mathbb{E} \int_{t_0}^{T} \mathcal{O}_1W^d_0(\xi)d\xi \leq 0$. Let $t \to t_0$, we then have $\mathcal{O}_1\phi(W_0) \leq 0$. This completes the proof.

**Sketched Proof of theorem 2.** The proofs follows the arguments used in Section 5 of Ishii and Lions (1990) and Theorem 3 of Davis et al. (1993), so we only present the main steps. First, we can construct a positive strict supersolution $h$ of (15) on $[0, T] \times \mathcal{T}_K$, by replacing the diffusion component with the corresponding jump-diffusion part in the proof of Theorem 3 of Davis et al. (1993). Then we define a linear combination of supersolutions $v$ and $h$, $v^\theta = \theta v + (1 - \theta)h$, where $\theta \in (0, 1)$, which is a viscosity supersolution of the equation $H(W, v_t, Dw, D^2v) - (1 - \theta)f$. Then based on arguments similar to Lemma 2 of Davis et al. (1993), we get $u \leq v^\theta$. Letting $\theta \to 1$ yields $u \leq v$.

**Sketched Proof of theorem 3.** Denote the solution derived in Section 3.2 as $\tilde{h}^{i, \delta}$. For convenience, we omit the superscript $i$ of $h^i$ and $\tilde{h}^{i, \delta}$ in the sequel. Define $h^\delta(s, z, x)$ such that $h^\delta(s, z, x) = \tilde{h}^\delta(s, z, x^\delta)$, for $s \in (s_i, s_{i-1})$, $z \in [z_j - \frac{1}{2}\sqrt{\delta}, z_j + \frac{1}{2}\sqrt{\delta})$, $x \in [x_q - \frac{1}{2}\epsilon, x_q + \frac{1}{2}\epsilon]$, and $h^\delta(s, z, x) = \exp\{-\gamma KA(z, x)\}$, for $s = 0$. Let

$$h(s, z, x) = \liminf_{\delta, \epsilon \to 0} \tilde{h}^\delta(s, z, x), \quad \bar{h}(s, z, x) = \limsup_{\delta, \epsilon \to 0} \tilde{h}^\delta(s, z, x).$$

We first show $\bar{h}$ is a viscosity supersolution of (23), that is, min $\{\mathcal{P}_3\phi, -\mathcal{Q}_3\phi, \mathcal{O}_3\phi\} \geq 0$. Suppose that $w_0$ is a local minimum of $\bar{h} - \phi$, on the domain for a smooth function $\phi$. That is, $\bar{h}(v_0) = \phi(v_0)$, $\phi$ is bounded outside a neighborhood of $v_0$ such that $\bar{h} \geq \phi(v)$. By definition, there exist sequences $\delta_n \in \mathbb{R}^+$ and $\gamma_n := (s_i^{\delta_n}, z_j^{\delta_n}, x_q^{\delta_n})$, such that $\delta_n \to 0,$
\(v_n \rightarrow v_0 := (s, z, x), \tilde{h}^{\delta_n}(v_n) \rightarrow h(v_0), v_n \) is a global minimum point of \(\psi_h := \tilde{h}^{\delta_n} - \phi\). Then \(\psi_h \rightarrow 0\) and \(\tilde{h}^{\delta_n}(v) \geq \phi(v) + \psi_h(v)\) for any \(v\).

For the first operator \(P_3\), the boundary constraint (31) implies that \(\tilde{h}^{\delta_n}(w_{\delta_n}) = \tilde{h}^{\delta_n}(w_{\delta_n})\), where \(w_{\delta_n}\) are the points on the buy boundary. Then we have \(\tilde{h}^{\delta_n}(w_{\delta_n}) \leq \phi(w_{\delta_n}) + \tilde{h}^{\delta_n}(w_{\delta_n}) - \phi(w_{\delta_n})\), which further implies that \(P_3\phi \geq 0\). Similar arguments can show that \(-Q_3\phi \geq 0\). For the third operator \(O_3\), we notice that numerical scheme (34) implies that \(\tilde{h}^{\delta_n}(w_{\delta_n}) = \mathbb{E}\{\tilde{h}^{\delta_n}(w_{\delta_{n+1}})\}\). Then \(\tilde{h}^{\delta_n}(w_{\delta_n}) \leq \mathbb{E}\{\phi(w_{\delta_{n+1}})\} + \tilde{h}^{\delta_n}(w_{\delta_n}) - \phi(w_{\delta_{n+1}})\), together with Itô’s formula, suggests that \(O_3\phi \geq 0\).

Similar arguments can be applied to show that \(\bar{h}\) is a viscosity subsolution of equation (23). Then the uniqueness of the solution indicates that \(\bar{h} \geq h\). By the definition of \(\bar{h}\) and \(\tilde{h}\), same argument implies that \(\bar{h} \leq h\). Finally, it suggests that \(\bar{h} = h = \underline{h}\) and the uniform convergence of \(h^{\delta}\) to \(h\).