3. Some Linear Time-Series Models

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Outline Stationarity ACFs Noises RW MA AR ARMA ARIMA ARFIMA Linear

Outline

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Stochastic processes and their properties

- A stochastic process can be described as a statistical phenomenon that evolves in time according to probabilistic laws. Mathematically, a stochastic process is a collection of random variables that are ordered in time and defined at a set of time points, which may be continuous or discrete.
- Most statistical problems are concerned with estimating the properties of a population from a sample.
- In time-series analysis, the order of observations is determined by time and it is usually impossible to make more than one observation at any given time.
- We may regard the observed time series as just one example of the infinite set of time series that might have been observed. This infinite set of time series is called the ensemble, and every member of the ensemble is a possible realization of the stochastic process.

Stochastic processes and their properties

A simple way of describing a stochastic process is to give the moments of the process. Denote the random variable at time t by $X(t)$ if time is continuous, and by X_t if time is discrete.

- Mean: The mean function $\mu(t)$ is defined for all t by

$$\mu(t) = E\big[X(t)\big]$$

- Variance: The variance function $\sigma^2(t)$ is defined for all t by

$$\sigma^2(t) = \mathrm{Var}\big[X(t)\big]$$

- Autocovariance: We define the acv.f. $\gamma(t_1, t_2)$ to e the covariance of $X(t_1)$ with $X(t_2)$,

$$\gamma(t_1, t_2) = E\big\{\big[X(t_1) - \mu(t_1)\big]\big[X(t_2) - \mu(t_2)\big]\big\}$$

Stationary processes

- A time series is said to be strictly stationary if the joint distribution of $X(t_1), \ldots, X(t_k)$ is the same as the joint distribution of $X(t_1+\tau), \ldots, X(t_k+\tau)$ for all $t_1, \ldots, t_k, \tau$.
- Strict stationarity implies that for $k=1$

$$\mu(t) \equiv \mu, \qquad \sigma^2(t) \equiv \sigma^2;$$

  for $k=2$,

$$\gamma(\tau) = E\{[X(t)-\mu][X(t+\tau)-\mu]\} = \text{Cov}[X(t), X(t+\tau)],$$

  which is called the autocovariance coefficient at lag τ.
- The size of $\gamma(\tau)$ depends on the units in which $X(t)$ is measured. One usually standardizes the acv.f. to produce a function autocorrelation function (ac.f.), which is defined by

$$\rho(\tau) = \gamma(\tau)/\gamma(0).$$

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Stationary processes

- A process is called second-order stationary (or weakly stationary) if its mean is constant and its acv.f. depends only on the lag, so that

$$E[X(t)] = \mu$$

  and

$$\text{Cov}[X(t), X(t+\tau)] = \gamma(\tau)$$

- This weaker definition of stationarity will generally be used from now on.

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Some properties of the autocorrelation function

Suppose a stationary stochastic process $X(t)$ has mean μ, variance σ^2, acv.f. $\gamma(\tau)$ and ac.f. $\rho(\tau)$. Then

$$\rho(\tau) = \gamma(\tau)/\gamma(0) = \gamma(\tau)/\sigma^2, \qquad \rho(0) = 1$$

- The ac.f. is an even function of lag, so that $\rho(\tau) = \rho(-\tau)$.
- $|\rho(\tau)| \leq 1$.
- The ac.f. does not uniquely identify the underlying model.

Although a given stochastic process has a unique covariance structure, the converse is not in general true.

Purely random processes

- A discrete-time process is called a purely random process if it consists of a sequence of random variables, $\{Z_t\}$, which are mutually independent and identically distributed. We normally assume that $Z_t \sim N(0, \sigma_Z^2)$.
- The independence assumption means that

$$\gamma(k) = \text{Cov}(Z_t, Z_{t+k}) = \begin{cases} \sigma_Z^2 & k = 0 \\ 0 & k = \pm 1, \pm 2, \dots \end{cases}$$

$$\rho(k) = \begin{cases} 1 & k = 0 \\ 0 & k = \pm 1, \pm 2, \dots \end{cases}$$

- The process is strictly stationary, and hence weakly stationary.
- A purely random process is sometimes called white color, particularly by engineers.

Purely random processes

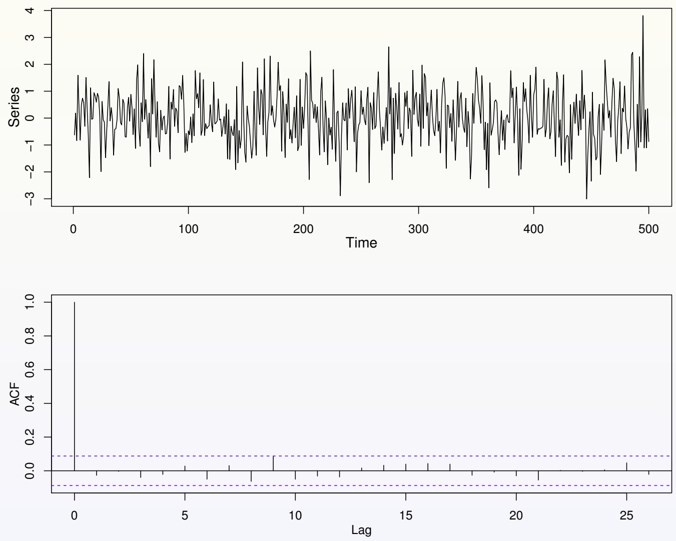

Figure 3.1: A purely random process with $\sigma_Z^2 = 1$ (top) and its correlogram (bottom).

Portmanteau tests

Random walks

- Suppose that $\{Z_t\}$ is a discrete-time, purely random process with mean μ and variance σ_Z^2. A process $\{X_t\}$ is said to be a random walk if $X_t = X_{t-1} + Z_t$.
- The process is customarily started at zero when $t = 0$, so that $X_t = \sum_{i=1}^{t} Z_i$.
- We find that $E(X_t) = t\mu$ and that $\mathrm{Var}(X_t) = t\sigma_Z^2$. As the mean and the variacne change with t, the process is non-stationary.
- The first differences of a random walk $\nabla X_t = X_t - X_{t-1} = Z_t$ form a purely random process, which is stationary.
- A good example of time series, which behaves like random walks, are share prices on successive days.

Random walks

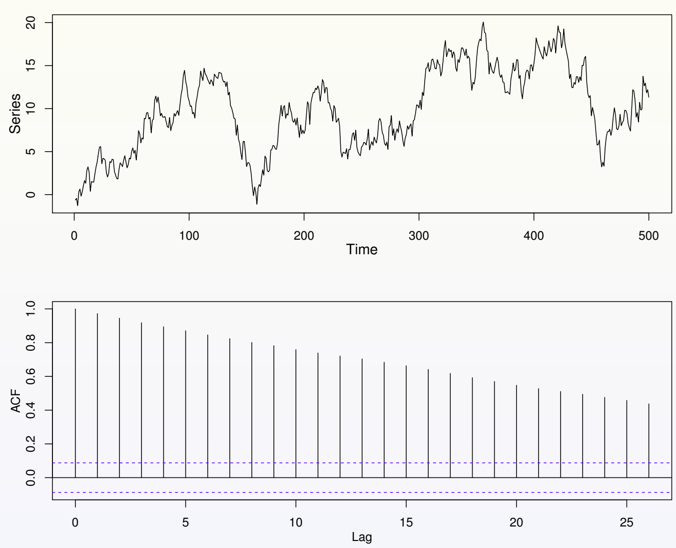

Figure 3.2: Simulated random walk (top) and its correlogram (bottom). The random walk series is generated from the white noise series in Figure 3.1.

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Moving average processes: MA(q) models

- A process $\{X_t\}$ is said to be a moving average process of order q (or MA(q) process) if

$$X_t = \beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}, \tag{1}$$

where $\{\beta_t\}$ are constants and Z_t is a purely random process with mean 0 and variacne σ_Z^2. The Zs are usually scaled so that $\beta_0 = 1$.
- We can show that, since Z_t's are independent, $E(X_t) = 0$, and Var$(X_t) = \sigma_Z^2 \sum_{i=0}^{q} \beta_i^2$.
- Using Cov$(Z_s, Z_t) = \sigma_Z^2 1_{\{s=t\}}$, we have, for $k \geq 0$,

$$\begin{aligned}\gamma(k) &= \text{Cov}(\beta_0 Z_t + \cdots + \beta_q Z_{t-q}, \beta_0 Z_{t+k} + \cdots + \beta_q Z_{t+k-q}) \\ &= \begin{cases} 0 & k > q \\ \sigma_Z^2 \sum_{i=0}^{q-k} \beta_i \beta_{i+k} & k = 0, 1, \ldots, q. \end{cases}\end{aligned}$$

As $\gamma(k)$ does not depend on t, and the mean is constant, the process is second-order stationary for all values of the $\{\beta_i\}$.

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MA(q) models: ACFs

- The ac.f. of the above MA(q) process is given by, for $k \geq 0$,

$$\rho(k) = \begin{cases} 1 & k = 0 \\ \sum_{i=0}^{q-k} \beta_i \beta_{i+k} \Big/ \sum_{i=0}^{q} \beta_i^2 & k = 1, \ldots, q \\ 0 & k > q \end{cases} \tag{2}$$

- Note that the ac.f. 'cuts off' at lag q, which is a special feature of MA processes.

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Example

Consider the series defined by rthe equation

$$X_t = Z_t + \theta Z_{t-1}, \qquad t = 0, \pm 1, \dots$$

where Z_t are independent Normal random variables with mean 0 and variance σ^2. we then have

$$\gamma_X(t+h,t) = \begin{cases} \sigma^2(1+\theta^2) & \text{if } h = 0, \\ \sigma^2\theta, & \text{if } h = \pm 1, \\ 0, & \text{if } |h| > 1. \end{cases}$$

Hence $\{X_t\}$ is stationary. The ac.f of $\{X_t\}$ is

$$\rho_X(h) = \begin{cases} 1 & \text{if } h = 0, \\ \theta/(1+\theta^2), & \text{if } h = \pm 1, \\ 0, & \text{if } |h| > 1. \end{cases}$$

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Example

Consider the series defined by the equation

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}, \qquad t = 0, \pm 1, \dots$$

where Z_t are independent Normal random variables with mean 0 and variance σ^2. Compute its ACFs.

Hints: Compute the autocovariance $\gamma(k) =$Cov(X_t, X_{t+k}) for $k = 0, 1, 2$ and $k \geq 3$, respectively.

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Example

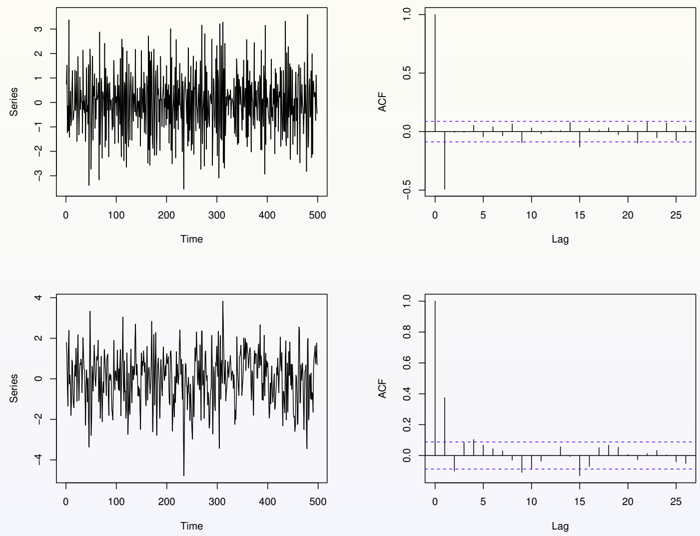

Figure 3.3: Simulated MA(1) (top left) and MA(2) (bottom left) processes and their corresponding correlograms (top right and bottom right).

Invertibility of an MA(q) process I

- Although there are no restrictions on the $\{\beta\}$ for a (finite-order) MA process to be stationary, restrictions are usually imposed on the $\{\beta_i\}$ to ensure that the process satisfies a condition called invertibility.
- Example: Consider the following MA(1) processes:

$$(A): X_t = Z_t + \theta Z_{t-1} \qquad (B): X_t = Z_t + \frac{1}{\theta} Z_{t-1}$$

We can show that (A) and (B) have exactly the same ac.f., hence we cannot identify an MA process uniquely from a given ac.f.

Invertibility of an MA(q) process II

- If we 'invert' models (A) and (B) by expressing Z_t in terms of X_t's, we have
$$(A): Z_t = X_t - \theta X_{t-1} + \theta^2 X_{t-2} - \dots$$
$$(B): Z_t = X_t - \frac{1}{\theta} X_{t-1} + \frac{1}{\theta^2} X_{t-2} - \dots$$
Note that the series of coefficients of X_{t-j} for models (A) and (B) cannot be convergent at the same time.
- In general, a process $\{X_t\}$ is said to be invertible if the random disturbance at time t (or innovations) can be expressed as a convergent sum of present and past values of X_t in the form
$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$$
where $\sum_{j=0}^{\infty} |\pi| < \infty$.

Invertibility of an MA(q) process III

- The definition above implies that an invertible process can be rewritten in the form an autoregressive process (possibly of infinite order), whose coefficients form a convergent sum.
- From the definition above, model (A) is said to be invertible whereas model (B) is not. — The imposition of the invertibility condition ensures that there is a unique MA process for a given ac.f.
- The invertibility condition for an MA process of any order can be expressed by using the backward shift operator, denoted by B, which is defined by
$$B^j X_t = X_{t-j} \qquad \text{for all } j.$$

Invertibility of an MA(q) process IV

- Denote $\theta(B) = \beta_0 + \beta_1 B + \cdots + \beta_q B^q$, then

$$X_t = \beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q} \Leftrightarrow X_t = \theta(B) Z_t$$

  It can be shown that an MA(q) process is invertible if the roots of the equation

$$\theta(B) = \beta_0 + \beta_1 B + \cdots + \beta_q B^q = 0$$

  all lie outside the unit circle, where B is regarded as a complex variable instead of as an operator.
- Example: In the MA(1) process $X_t = Z_t + \theta Z_{t-1}$, $\theta(B) = 1 + \theta B$ and the invertibility condition is $|\theta| < 1$.

How does the invertibility condition work? I

- In the first-order case for model A, we have $\theta(B) = 1 + \theta B$, which has root $B = -1/\theta$. Provided that $|\theta| < 1$, the root $B = -1/\theta$ is real and lies "outside the unit circle". So again we see that model A is invertible if $|\theta| < 1$. Furthermore, if we regard B as a complex variable, the operator $1/\theta(B)$ can be expanded as

$$\frac{1}{1+\theta B} = 1 + \sum_{i=1}^{\infty} (-\theta)^i B^i. \tag{3}$$

  When $|\theta| < 1$, this infinite series is convergent since

$$|1 + \sum_{i=1}^{\infty} (-\theta)^i| \leq 1 + \sum_{i=1}^{\infty} |\theta|^i = \frac{1}{1-|\theta|}.$$

How does the invertibility condition work? II

Hence

$$Z_t = \frac{1}{\theta(B)} X_t = \Big(1 + \sum_{i=1}^{\infty} (-\theta)^i B^i\Big) X_t = X_t + \sum_{i=1}^{\infty} (-\theta)^i X_{t-i},$$

and thus $\{X_t\}$ is invertible.

- The argument above can be extended to MA(q) processes. Suppose that $\theta(B)$ can be decomposed as the following form

$$\theta(B) = (1 + \theta_1 B) \cdots (1 + \theta_q B),$$

where $\theta_1, \ldots, \theta_q$ could possibly take complex values. Then the operator $1/\theta(B)$ can be written as

$$\frac{1}{\theta(B)} = \prod_{j=1}^{q} \frac{1}{1 + \theta_j B} = \prod_{j=1}^{q} \Big(1 + \sum_{i=1}^{\infty} (-\theta_j)^i B^i\Big). \tag{4}$$

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How does the invertibility condition work? III

When all the roots, $-1/\theta_1, \ldots, -1/\theta_q$, are outside the unit circle, the product of infinite series in (4) is convergent, and hence $Z_t = 1/\theta(B) X_t$ can be written in the form of (**??**), and therefore $\{X_t\}$ is invertible.

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Autoregressive processes

- A process $\{X_t\}$ is said to be an autoregressive process of order p (or AR(p)) if
$$X_t = \alpha_1 X_{t-1} + \cdots + \alpha_p X_{t-p} + Z_t, \tag{5}$$
where Z_t is a purely random process with mean 0 and variacne σ_Z^2.
- The above AR(p) process can be written as
$$(1 - \alpha_1 B - \cdots - \alpha_p B^p) X_t = Z_t.$$

An AR(1) process I

- Consider the first-order AR process
$$X_t = \alpha X_{t-1} + Z_t, \tag{6}$$
By successive substitution, we obtain that, if $|\alpha| < 1$,
$$\begin{aligned} X_t &= \alpha(\alpha X_{t-2} + Z_{t-1}) + Z_t = \alpha^2(\alpha X_{t-3} + Z_{t-2}) + \alpha Z_{t-1} + Z_t \\ &= \cdots = Z_t + \alpha Z_{t-1} + \alpha^2 Z_{t-2} + \ldots \end{aligned}$$

An AR(1) process II

- We can use the backward shift operator B to explore the duality between AR and MA process. Note that (6) can be written as

$$(1-\alpha B)X_t = Z_t,$$

so that

$$\begin{aligned} X_t &= Z_t/(1-\alpha B) = (1+\alpha B + \alpha^2 B^2 + \dots)Z_t \\ &= Z_t + \alpha Z_{t-1} + \alpha^2 Z_{t-2} + \dots \end{aligned}$$

An AR(1) process III

- The above form implies that

$$E(X_t) = 0, \qquad \text{Var}(X_t) = \sigma_Z^2(1+\alpha^2+\alpha^4+\dots)$$

If $|\alpha| < 1$, we have $\text{Var}(X_t) = \sigma_X^2 = \sigma_Z^2(1-\alpha^2)$, and for $k \geq 0$,

$$\begin{aligned} \gamma(k) &= E(X_t X_{t+k}) = E\big[\big(\sum \alpha^i Z_{t-i}\big)\big(\sum \alpha^j Z_{t+k-j}\big)\big] \\ &= \sigma_Z^2 \sum_{i=0}^{\infty} \alpha^i \alpha^{k+i} = \alpha^k \sigma_Z^2/(1-|\alpha|^2) \qquad \text{if } |\alpha| < 1 \\ &= \alpha^k \sigma_X^2 \end{aligned}$$

- We find that the process (6) is second-order stationary provided $|\alpha| < 1$.

An AR(1) process IV

- Since $\rho(k) = \gamma(k)/\gamma(0)$ and $\gamma(k) = \gamma(-k)$, we have

$$\rho(-k) = \rho(k) = \alpha^k \qquad k = 0, 1, 2, \ldots$$

  or

$$\rho(k) = \alpha^{|k|} \qquad k = 0, \pm 1, \pm 2, \ldots$$

- The acv.f. and ac.f. can also be written recursively

$$\gamma(k) = \alpha\gamma(k-1), \qquad \rho(k) = \alpha\rho(k-1) \qquad \text{for } k > 0.$$

- Examples of the ac.f. of the AR(1) process for different values of α.

Simulated AR(1) examples

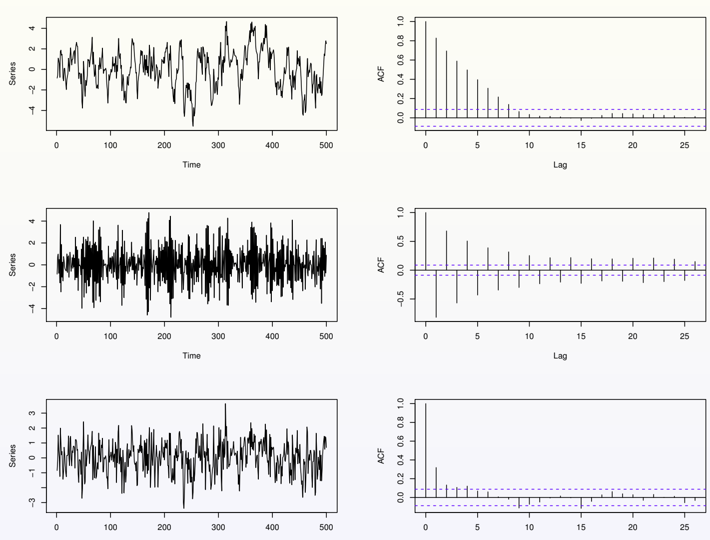

Figure 3.4: Three simulated AR(1) processes and their correlograms. Top: $X_t = 0.8X_{t-1} + Z_t$, $Z_t \sim N(0,1)$; Middle: $X_t = -0.8X_{t-1} + Z_t$, $Z_t \sim N(0,1)$; Bottom: $X_t = 0.3X_{t-1} + Z_t$, $Z_t \sim N(0,1)$.

General AR(p) process

Consider the AR(p) process

$$(1 - \alpha_1 B - \cdots - \alpha_p B^p) X_t = Z_t,$$

or equivalently as

$$X_t = Z_t / (1 - \alpha_1 B - \cdots - \alpha_p B^p) = f(B) Z_t \qquad (7)$$

where $f(B) = (1 - \alpha_1 B - \cdots - \alpha_p B^p)^{-1} = 1 + \beta_1 B + \beta_2 B^2 + \ldots$

- The relationship between the α's and β's can be found.
- The necessary condition for (7) to be stationary is that its variance or $\sum_i \beta_i^2$ converges. The acv.f. is given by

  $$\gamma(k) = \sigma_Z^2 \sum_{i=0}^{\infty} \beta_i \beta_{i+k} \qquad \text{for } \beta_0 = 1.$$

  A sufficient condition for this to converge, and hence for stationarity is that $\sum |\beta_i|$ converges.

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Yule-Walker equations

- Since $\{\beta_j\}$ might be algebriaclly hard to find, an alternative way is to assume the process is stationary, multiply through (5) by X_{t-k}, take expectations and divide by σ_X^2, asuming that Var$(X_t) < \infty$. Then using $\rho(-k) = \rho(k)$ for all k, we find the Yule-Walker equations

  $$\rho(k) = \alpha_1 \rho(k-1) + \cdots + \alpha_p \rho(k-p) \qquad \text{for all } k > 0. \qquad (8)$$

- This set of difference equations has the general solution

  $$\rho(k) = A_1 \pi_1^{|k|} + \cdots + A_p \pi_p^{|k|},$$

  where $\{\pi_i\}$ are the roots of the so-called auxiliary equation

  $$y^p - \alpha_1 y^{p-1} - \cdots - \alpha_p = 0.$$

- The constants $\{A_i\}$ are chosen to satisfy the initial conditions depending on $\rho(0) = 1$. The first $(p-1)$ Yule–Walker equations provide $(p-1)$ further restrictions on the $\{A_i\}$ using $\rho(0) = 1$ and $\rho(k) = \rho(-k)$.

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Statoinary conditions

- From the general form of $\rho(k)$, it is clear that $\rho(k)$ tends to zero as k increases provided that $|\pi_i| < 1$ for all i, and this is a necessary and sufficient condition for the AR(p) process to be stationary.
- An equivalent way of expressing the stationarity condition is to say that the roots of the equation

$$\phi(B) = 1 - \alpha_1 B - \cdots - \alpha_p B^p = 0 \tag{9}$$

must lie outside the unit circle (where we again regard B as a complex variable, rather than as an operator, so that the roots, which may be complex, are greater than one in modulus).

- For $p = 2$, the Yule-Walker equation (8) is a set of difference equations and has the general solution

$$\rho(k) = A_1 \pi_1^{|k|} + A_2 \pi_2^{|k|},$$

where $\{\pi_i\}$ are the roots of the so-called auxiliary equation

$$y^2 - \alpha_1 y - \alpha_2 = 0.$$

- Important Example: Computing the ACFs of the AR(2) process
Suppose π_1, π_2 are the roots of the quadratic equation

$$y^2 - \alpha_1 y - \alpha_2 = 0.$$

Here $|\pi_i| < 1$ if

$$\left| \frac{\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_2}}{2} \right| < 1$$

from which we can show that the stationarity region is the triangular region satisfying

$$\alpha_1 + \alpha_2 < 1, \qquad \alpha_1 - \alpha_2 > -1, \qquad \alpha_2 > -1$$

- When the roots are real, we have $\rho(k) = A_1\pi_1^{|k|} + A_2\pi_2^{|k|}$ where the constants A_1, A_2 are also real and may be found as follows. Since $\rho(0) = 1$, we have

$$A_1 + A_2 = 1$$

while the first Yule-Walker equation gives

$$\rho(1) = \alpha_1\rho(0) + \alpha_2\rho(-1) = \alpha_1 + \alpha_2\rho(1) \Rightarrow \rho(1) = \alpha_1/(1-\alpha_2).$$

Hence we find

$$A_1 = \frac{\alpha_1/(1-\alpha_2) - \pi_2}{\pi_1 - \pi_2}, \qquad A_2 = 1 - A_1$$

Remark: We only considered process with mean zero here, but non-zero means can be dealt with by rewritting (**??**) in the form

$$X_t - \mu = \alpha_1(X_{t-1} - \mu) + \cdots + \alpha_p(X_{t-p} - \mu) + Z_t,$$

this does not affect the ac.f.

Example I

Example 1 Consider the AR(2) process given by

$$6X_t = -X_{t-1} + X_{t-2} + Z_t,$$

is this process stationary? If so, what is its ac.f.?
Solution. The roots of $1 + \frac{1}{6}B - \frac{1}{6}B^2 = 0$ are -2 and 3, they are outside the unit circle, hence X_t is stationary. Note that the roots of $y^2 + \frac{1}{6}y - \frac{1}{6} = 0$ are -1/2 and $1/3$, the ACFs of this process are given by

$$\rho(k) = A_1\Big(-\frac{1}{2}\Big)^{|k|} + A_2\Big(\frac{1}{3}\Big)^{|k|}, \quad k = 0, 1, \dots,$$

Since $\rho(0) = 1 = A_1 + A_2$, and $\rho(1) = -\rho(0) + 6\rho(-1)$ gives $\rho(1) = -1/5 = -\frac{A_1}{2} + \frac{A_2}{3}$, we have $A_1 = 16/25$ and $A_2 = 9/25$.

Example II

Example 2 Consider the AR(2) process given by

$$X_t = X_{t-1} - \frac{1}{2}X_{t-2} + Z_t,$$

is this process stationary? If so, what is its ac.f.?
Solution. The roots of equation $\phi(B) = 1 - B + \frac{1}{2}B^2 = 0$ are complex, namely, $1 \pm i$, whose modulus both exceeds one, hence the process is stationary. To calculate the ac.f. of the process, we use the first Yule-Walker equation to give $\rho(1) = \rho(0) - \frac{1}{2}\rho(-1) = 1 - \frac{1}{2}\rho(1)$, which yields $\rho(1) = 2/3$. For $k \geq 2$, the Yule-Walker equations are $\rho(k) = \rho(k-1) - \frac{1}{2}\rho(k-2)$, which indicates the auxiliary equation $y^2 - y + \frac{1}{2} = 0$ with roots $y = (1 \pm i)/2 = e^{\pm i\pi/4}/\sqrt{2}$. Using $\rho(0) = 1$ and $\rho(1) = 2/3$, some algebra gives

$$\rho(k) = 2^{-k/2}\Big(\cos\frac{\pi k}{4} + \frac{1}{3}\sin\frac{\pi k}{4}\Big), \qquad k = 0, 1, 2, \ldots$$

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Example III

Example 3 Consider the AR(2) process given by

$$12X_t = -X_{t-1} + X_{t-2} + Z_t,$$

is this process stationary? If so, what is its ac.f.?
Solutions: (1) The roots of $12 + B - B^2 = 0$ are 4 and -3, which are outside the unit circle, hence X_t is stationary. (2) The ACFs of X_t are

$$\rho(k) = A_1\Big(\frac{1}{4}\Big)^k + A_2\Big(-\frac{1}{3}\Big)^k, \qquad k \geq 0,$$

where $A_1 + A_2 = \rho(0) = 1$ and $\frac{A_1}{4} - \frac{A_2}{3} = \rho(1) = \rho(1) = -\frac{1}{11}$, hence $A_1 = \frac{32}{77}$ and $A_2 = \frac{45}{77}$.
Example 4 Compute the ac.f. of the AR(2) process

$$X_t = \frac{1}{6}X_{t-1} + \frac{1}{6}X_{t-2} + Z_t.$$

Solutions: $\rho(h) = \frac{4}{25}(\frac{1}{2})^h + \frac{21}{25}(-\frac{1}{3})^h$, $h = 0, 1, 2, \ldots$

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Example

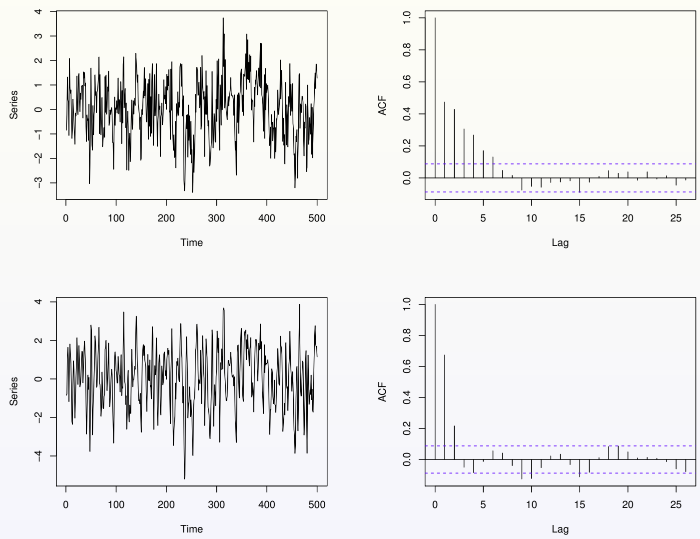

Figure 3.5: Two simulated AR(2) processes and their correlograms. Top: $X_t = \frac{1}{3}X_{t-1} + \frac{2}{9}X_{t-2} + Z_t$, $Z_t \sim N(0,1)$; Bottom: $X_t = X_{t-1} - \frac{1}{2}X_{t-2} + Z_t$, $Z_t \sim N(0,1)$

ARMA(p,q) models

- A mixed autoregressive/moving-average process containing p AR terms and q MA terms is said to be an ARMA process of order (p,q). It is given by

$$X_t - \alpha_1 X_{t-1} - \dots \alpha_p X_{t-p} = Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}. \quad (10)$$

- Let $\phi(B) = 1 - \alpha_1 B - \cdots - \alpha_p B^p$, $\theta(B) = 1 + \beta_1 B + \cdots + \beta_q B^q$. Equation (10) may be written in the form

$$\phi(B)X_t = \theta(B)Z_t.$$

- The condition on the model parameters to make the process stationary and invertible are the same as for a pure AR or MA process.

Stationarity and invertibility conditions I

- The conditions on the model parameters to make the process stationary and invertible are the same as for a pure AR or pure MA process.
- The values of $\{\alpha_i\}$, which make the process stationary, are such that the roots of
$$\phi(B) = 0$$
lie outside the unit circle.
- The values of $\{\beta_i\}$, which make the process invertible, are such that the roots of
$$\theta(B) = 0$$
lie outside the unit circle.

Outline Stationarity ACFs Noises RW MA AR ARMA ARIMA ARFIMA Linear

Stationarity and invertibility conditions II

- The importance of ARMA processes lies in the fact that a stationary time series may often be adequately modelled by an ARMA model involving fewer parameters than a pure MA or AR process by itself.
- The ac.f. of the general ARMA process can be found using similar procedures as for AR processes. First, multiply through Equation (10) by X_{t-k} and take expectations. Note that, for $k \geq q+1$, $Z_t, \ldots, Z_{t-q}$ are independent of X_{t-k}. Hence the expected values of $Z_t X_{t-k}, \ldots, Z_{t-q} X_{t-k}$ are all zero. If $k \geq p$, we can further divide both sides by $\gamma(0)$. Then we find the following Yule-Walker equations for general ARMA(p, q) processes
$$\rho(k) = \alpha_1 \rho(k-1) + \cdots + \alpha_p \rho(k-p), \quad k \geq \max(p, q+1). \quad (11)$$
- Note that (11) has the same form as that of an AR process except that their initial conditions are different. The initial conditions of the Yule-Walker equations are usually computed seperately.

Example: ACFs of ARMA(p,q) models I

$$X_t = \alpha X_{t-1} + Z_t + \beta Z_{t-1}, \qquad |\alpha| < 1, |\beta| < 1.$$

- The Yule-Walker equations are $\rho(k) = \alpha\rho(k-1)$, $k \geq 2$.
- We can show that

$$\gamma(1) = \alpha\gamma(0) + \beta\sigma_Z^2, \qquad \gamma(0) = \alpha^2\gamma(0) + (1 + 2\alpha\beta + \beta^2)\sigma_Z^2.$$

- We then obtain that

$$\gamma(0) = \frac{1 + 2\alpha\beta + \beta^2}{1-\alpha^2}\sigma_Z^2, \qquad \gamma(1) = \frac{(1+\alpha\beta)(\alpha+\beta)}{1-\alpha^2}\sigma_Z^2.$$

- Therefore,

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{(1+\alpha\beta)(\alpha+\beta)}{1+2\alpha\beta+\beta^2}.$$

Outline Stationarity ACFs Noises RW MA AR ARMA ARIMA ARFIMA Linear

Example: ACFs of ARMA(p,q) models II

Consider the ARMA(2,1) model $X_t = \alpha X_{t-2} + Z_t + \beta Z_{t-1}$. (a) Under which condition, X_t becomes stationary and invertiable? (b) Compute the ACFs of the process.

Sketch of the solution:

- When $0 < \alpha < 1$ and $\beta \in (-1, 0) \cup (0, 1)$, X_t is stationary and invertible.
- From Yule-Walker equation, we have $\rho(k) = \alpha\rho(k-2)$ for $k \geq 2$. We notice the equations $\gamma(0) = \alpha^2\gamma(0) + (1+\beta^2)\sigma^2$ and $\gamma(1) = \alpha\gamma(1) + \beta\sigma^2$, and then obtain $\rho(1) = [(1+\alpha)\beta]/(1+\beta^2)$.

Question: How to compute the ACFs of the ARMA(1,2) process: $X_t = \alpha X_{t-1} + Z_t - \beta Z_{t-2}$?

Haipeng Xing, State University of New York, Stony Brook

Example

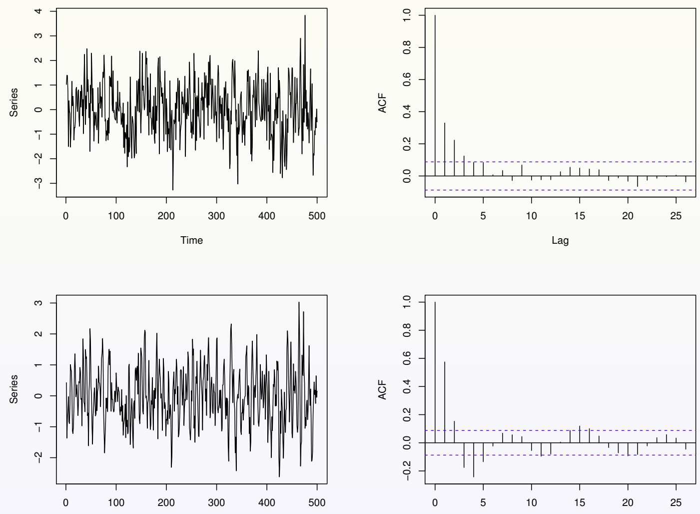

Figure 3.6: Two simulated ARMA processes and their correlograms. Top: $X_t = 0.7X_{t-1} + Z_t - 0.4Z_{t-1}$, $Z_t \sim N(0,1)$; Bottom: $X_t = 0.9X_{t-1} - 0.5X_{t-2} + Z_t - 0.2Z_{t-1} + 0.25Z_{t-2}$, $Z_t \sim N(0, 0.5)$.

AR and MA representations

- An ARMA model can be expressed as a pure MA process in the form $X_t = \psi(B)Z_t$, where $\psi(B) = \sum \psi_i B^i$ is the MA operator, which may be of infinite order. By comparison, we see that $\psi(B) = \theta(B)/\phi(B)$.
- An ARMA model can also be expressed as a pure AR process in the form $\pi(B)X_t = Z_t$, where $\pi(B) = \phi(B)/\theta(B)$. By convention we write $\pi(B) = 1 - \sum_{i\geq 1} \pi_i B^i$ so that

$$X_t = \sum_{i=1}^{\infty} \pi_i X_{t-i} + Z_t.$$

Example Find the ψ weights and π weights for the ARMA(1,1) process given by

$$X_t = 0.5X_{t-1} + Z_t - 0.3Z_{t-1}.$$

Here $\phi(B) = 1 - 0.5B$ and $\theta(B) = 1 - 0.3B$. We can show that the process is stationary and invertible. Then

$$\begin{aligned}\psi(B) = \theta(B)/\phi(B) &= (1-0.3B)(1-0.5B)^{-1} \\ &= (1-0.3B)(1+0.5B+0.5^2B^2+\dots) \\ &= 1+0.2B+0.1B^2+0.005B^3+\dots\end{aligned}$$

Hence

$$\psi_i = 0.2 \times 0.5^{i-1} \qquad \text{for } i = 1, 2, \dots$$

Similarly we find

$$\pi_i = 0.2 \times 0.3^{i-1} \qquad \text{for } i = 1, 2, \dots$$

ARIMA(p, d, q) models

- If the observed time series is nonstationary in the mean, then we can difference the series, as suggested in Chapter 2.
- If X_t is replaced by $\nabla^d X_t$ in (10), then we have a model capable of describing certain types of non-stationary series. Such a model is called an 'integrated' model.
- Let $W_t = \nabla^d X_t = (1-B)^d X_t$, the general ARIMA (autoregressive integrated moving average) process is of the form

$$W_t - \alpha_1 W_{t-1} - \dots - \alpha_p W_{t-p} = Z_t + \dots + \beta_q Z_{t-q} \tag{12}$$

or

$$\phi(B)W_t = \theta(B)Z_t,$$

where $\phi(B)$ and $\theta(B)$ are defined in (10).

- We may write (12) in the form

$$\phi(B)(1-B)^d X_t = \theta(B) Z_t, \tag{13}$$

the model above is said to be an ARIMA process of order (p, d, q).

- The model for X_t is clearly non-stationary, as the AR operator $\phi(B)(1-B)^d$ has d roots on the unit circle. Note that the random walk can be regarded as an ARIMA(0,1,0) process.
- ARIMA models can be extended to include seasonal effects, as discussed later.

Example: ACFs of ARIMA models

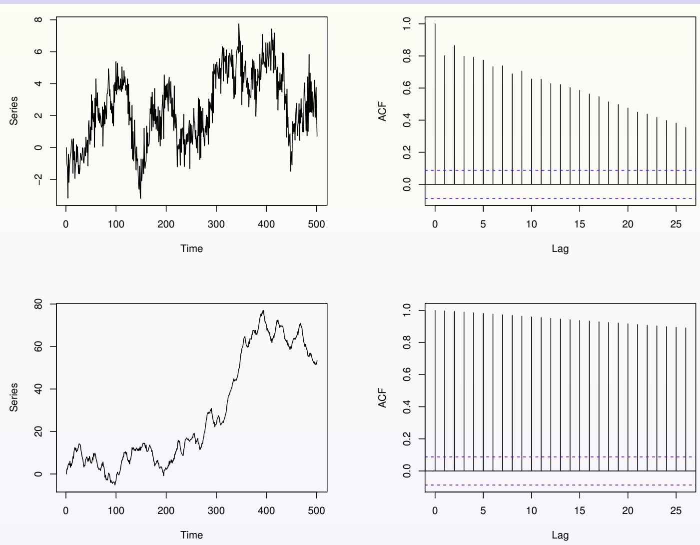

Figure 3.7: Two simulated ARIMA processes and their correlograms. Top: $(1+0.5B)(1-B)X_t = (1+0.3B)Z_t$; Bottom: $(1-0.6B)(1-B)X_t = (1-0.3B+0.5B^2)Z_t$.

Fractional differencing and long-memory models

- An interesting variant of ARIMA modelling arises with the use of what is called fractional differencing, leading to a fractional integrated ARMA (abbreviated to ARFIMA) model.
- When d is not an integer, then the dth difference $(1-B)^d X_t$ becomes a fractional difference, and may be represented by its binomial expansion, namely

$$(1-B)^d X_t = \left[1 - dB + \frac{d(d-1)}{2!}B^2 - \frac{d(d-1)(d-2)}{3!}B^3 + \dots\right] X_t.$$

- It can be shown that an ARFIMA process is stationary provided that $-0.5 < d < 0.5$.
- A stationary ARFIMA model, with $0 < d < 0.5$, is of particular interest as such a process is not only stationary, but is also an example of what is called a long-memory model.

Fractional differencing and long-memory models

- A stationary process with ac.f. $\rho(k)$ is said to be a long-memory process if $\sum_{k=0}^{\infty} |\rho(k)|$ does not converge. In particular, the latter condition applies when the ac.f. $\rho(k)$ is of the form $\rho(k) \sim Ck^{2d-1}$ as $k \to \infty$, where C is a constant, not equal to zero, and $0 < d < 0.5$.
- It can be shown that a stationary ARFIMA model, with differencing parameter d in the range $0 < d < 0.5$, has an ac.f. $\rho_k \sim Ck^{2d-1}$as $k \to \infty$, where C is a constant. Hence, ARFIMA(d) is a long-memory process.

Example

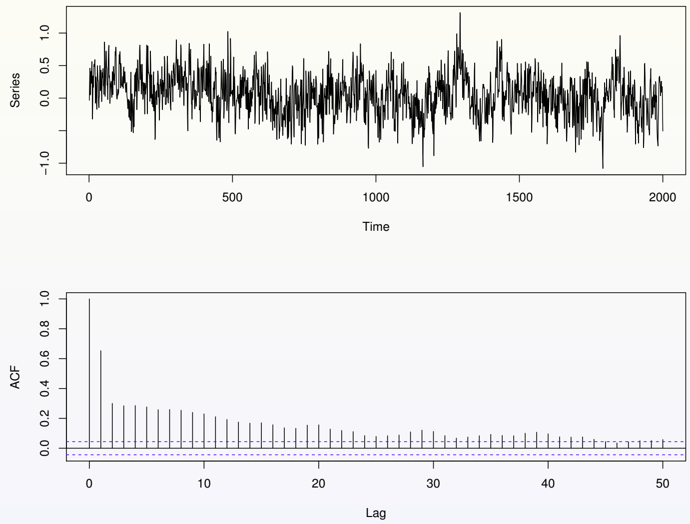

Figure 3.8: A simulated ARFIMA(1, $d = 0.4$, 2) process, $(1-0.3B)(1-B)^d X_t = (1-0.3B+0.5B^2)Z_t$ with $Z_t \sim N(0, 0.04)$, and its correlogram.

General linear processes

- A general class of processes may be written as an MA process, of possibly infinite order, in the form

$$X_t = \sum_{i=0}^{\infty} \psi_i Z_{t-i}. \tag{14}$$

  where $\sum_{i=0}^{\infty} |\psi_i| < \infty$ to that the process is stationary. A stationary process defined by (14) is called a general linear process.

- Stationary AR and ARMA processes can also be expressed as a general linear process using the duality between AR and MA processes.

The Wold decomposition theorem

- The Wold decomposition theorem: Any discrete-time stationary process can be expressed as the sum of two uncorrelated processes, one purely deterministic and one purely indeterministic, which are defined as follows.
- Regress X_t on $(X_{t-q}, X_{t-q-1}, \dots)$ and denote the residual variance from the resulting linear regression model by τ_q^2. If $\lim_{q\to\infty} \tau_q^2 = 0$, then the process can be forecast exactly, we say that $\{X_t\}$ is purely deterministic.
- If $\lim_{q\to\infty} \tau_q^2 = \text{Var}(X_t)$, then the linear regression on the remote past is useless for prediction purposes, and we say that $\{X_t\}$ is purely indeterministic.
- The Wold decomposition theorem also says that the purely indeterministic component can be wrttien as a linear sum of a sequence of uncorrelated random variables, say $\{Z_t\}$.