

Forecasting and Box-Jenkins Methodology

Outline

- 1 Mean square error prediction
- 2 Box-Jenkins procedure

Mean square error prediction

Suppose that the observations X_1, \dots, X_n follow some time series model. Denote $\mathcal{F}_n = (X_1, \dots, X_n)$. We want to find an estimate $f(X_1, \dots, X_n)$ for X_{n+k} ($k \geq 1$), i.e., a k -step ahead forecast at forecast origin n . A common used forecast criterion is to consider minimizing the **mean square error** (MSE) of your forecast, i.e.,

$$\begin{aligned} & \min_f E\{[X_{n+k} - f(X_1, \dots, X_n)]^2 | \mathcal{F}_n\} \\ &= \text{Var}(X_{n+k} | \mathcal{F}_n) + \min_f E\{[E(X_{n+k} | \mathcal{F}_n) - f(X_1, \dots, X_n)]^2 | \mathcal{F}_n\}. \end{aligned}$$

The above MSE is minimized by choosing $\hat{f} = E(X_{n+k} | \mathcal{F}_n)$. Therefore, we refer to $E(X_{n+k} | \mathcal{F}_n)$ as the **k -step ahead forecast**.

Prediction error

- The k -step ahead prediction error is defined as

$$e_n(k) = X_{n+k} - E(X_{n+k} | \mathcal{F}_n),$$

which is random variable as X_{n+k} is not observed.

- To measure the prediction error, we usually compute the **variance of the prediction error**:

$$\text{Var}(e_n(k)) = \text{Var}(X_{n+k} | \mathcal{F}_n).$$

- Based on the predictor and its error variance, we may construct prediction intervals. For instance, if X_t 's are normally distributed, we could consider its 95% prediction interval

$$E(X_{n+k} | \mathcal{F}_n) \pm z_{0.975} \sqrt{\text{Var}(X_{n+k} | \mathcal{F}_n)}.$$

Mean square error prediction

Given observations X_1, \dots, X_n , compute the k -step ($k \geq 1$) ahead prediction for the following time series models.

Example [Purely random process] X_t i.i.d. $(0, \sigma^2)$. The k -step ($k = 1, 2, \dots$) ahead prediction is given by

$$\hat{X}_n(k) = E(X_{n+k} | \mathcal{F}_n) = E(X_{n+k}) = 0,$$

Example [Random walk] $X_t = X_{t-1} + Z_t$, Z_t i.i.d. $(0, \sigma^2)$. The k -step ($k = 1, 2, \dots$) ahead prediction is given by

$$\hat{X}_n(k) = E(X_{n+k} | \mathcal{F}_n) = X_n + E(Z_{n+1} + \dots + Z_{n+k}) = X_n,$$

Mean square error prediction

Example [AR(p) process] $X_t = \alpha_1 X_{t-1} + \dots + \alpha_p X_{t-p} + Z_t$ with Z_t i.i.d. $(0, \sigma^2)$. The k -step ($k = 1, 2, \dots$) ahead prediction is given by

$$\hat{X}_n(k) = E(X_{n+k} | \mathcal{F}_n) = \sum_{i=1}^p \alpha_i E(X_{n+k-i} | \mathcal{F}_n),$$

where $E(X_{n+k-i} | \mathcal{F}_n) = X_{n+k-i}$ if $k \leq i$.

Example [MA(q) process] $X_t = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}$ with Z_t i.i.d. $(0, \sigma^2)$. The k -step ($k = 1, 2, \dots$) ahead prediction is given by

$$\hat{X}_n(k) = E(X_{n+k} | \mathcal{F}_n) = \sum_{i=1}^q \theta_i E(Z_{n+k-i} | \mathcal{F}_n),$$

where $E(Z_{n+k-i} | \mathcal{F}_n) = Z_{n+k-i}$, if $k \leq i$, and 0, if $k > i$.

Mean square error prediction

Example [ARMA(p, q) process]

$X_t = \alpha_1 X_{t-1} + \cdots + \alpha_p X_{t-p} + Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$ with Z_t i.i.d. $(0, \sigma^2)$. The k -step ($k = 1, 2, \dots$) ahead prediction is given by

$$\hat{X}_n(k) = E(X_{n+k} | \mathcal{F}_n) = \sum_{i=1}^p \alpha_i E(X_{n+k-i} | \mathcal{F}_n) + \sum_{i=1}^q \theta_i E(Z_{n+k-i} | \mathcal{F}_n),$$

where $E(X_{n+k-i} | \mathcal{F}_n) = X_{n+k-i}$ if $k \leq i$, and $E(Z_{n+k-i} | \mathcal{F}_n) = 0$, if $k \leq i$, and 0, if $k > i$.

Box-Jenkins procedure

Box and Jenkins summarized the following procedure to analyze an observed time series.

- ① Difference the series until stationarity.
- ② Identify the trend and seasonal components
- ③ Fit an ARMA model to the remainder series in (2).
- ④ Perform diagnostic analysis for residuals in (3). If the series is not well fitted, repeat (3).
- ⑤ Compute the k -step ahead forecast.