AMS316 Final in 2012 Solution

Assume that Z_t are independent and identically distributed normal random variables with mean 0 and variance σ^2.

1. Consider the process $X_t = a_0 + a_1 t + \cdots + a_k t^k + s_t + y_t$, in which s_t is a deterministic process satisfying $s_t = s_{t+d}$ and $s_t + \cdots + s_{t+d-1} = 0$, y_t is stationary process with mean zero.

   (a) What are the trend and seasonal components in the process X_t?

   (b) Let $s_t \equiv 0$ and consider the differenced series $Y_t = (1-B)^r X_t$. What is the minimal value of r to make Y_t stationary?

   Solution: (a) The trend component is $a_0 + a_1 t + \cdots + a_k t^k$ and the seasonal component is s_t. (b) Since each differencing operator $1 - B$ reduces the order of the polynomial part by 1, we need at least k differencing steps to make Y_t stationary.

2. Consider the stationary process $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$.

   (a) (8 points) Compute the lag-k autocorrelations $\rho(k)$ for $k \geq 1$.

   (b) (12 points) What are the 1-step and 2-step ahead forecasts of the process at forecast origin X_n? What are their corresponding forecasting error?

   Solution: (a) By definition of autocovariance functions, we have $\gamma(0) = \mathrm{Var}(X_t) = (1 + \theta_1^2 + \theta_2^2)\sigma^2$, $\gamma(1) = \mathrm{Cov}(X_t, X_{t-1}) = \theta_1(1+\theta_2)\sigma^2$, $\gamma(2) = \mathrm{Cov}(X_t, X_{t-2}) = \theta_2\sigma^2$ and $\gamma(k) = 0$ for $k \geq 3$. Hence we have

$$\rho(k) = \gamma(k)/\gamma(0) = \begin{cases} \dfrac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} & k = 1, \\ \dfrac{\theta_2}{1+\theta_1^2+\theta_2^2} & k = 2, \\ 0 & k \geq 3. \end{cases}$$

   (b) The 1- and 2-step ahead forecasts are $X_{n+1|n} = E(X_{n+1}|\mathcal{F}_n) = \theta_1 Z_n + \theta_2 Z_{n-1}$ and $X_{n+2|n} = E(X_{n+2}|\mathcal{F}_n) = \theta_2 Z_n$, and the correpsonding forecasting errors are $e_n(1) = X_{n+1} - X_{n+1|n} = Z_{n+1}$ and $e_n(2) = X_{n+2} - X_{n+2|n} = Z_{n+2} + \theta_1 Z_{n+1}$.

3. Consider the AR(2) process:

$$X_t - \frac{2}{a(a+2)} X_{t-1} - \frac{1}{a(a+2)} X_{t-2} = Z_t,$$

   in which the real number a is a parameter.

(a) (10 points) For what values of real number a, the process is stationary?

(b) (10 points) What is the autocorrelation of the process at lag k, i.e., $\rho(k)$? ($k \geq 0$).

(c) (10 points) What are the 1-step and 2-step ahead forecasts of the process at forecast origin X_n?

Solution: (a) The roots of

$$1 - \frac{2}{a(a+2)}B - \frac{1}{a(a+2)}B^2 = 0$$

are a and $-(a+2)$. The stationarity condition needs that $|a| > 1$ and $|a+2| > 1$. Simplifying these inequalities yields that $a > 1$ or $a < -3$. (b) The ACFs have the form

$$\rho(k) = A_1\left(\frac{1}{a}\right)^{|k|} + A_2\left(-\frac{1}{a+2}\right)^{|k|},$$

where $A_1 + A_2 = \rho(0) = 1$, $A_1/a - A_2/(a+2) = \rho(1)$. Note that Yule-Walker equation suggests that

$$\rho(1) - \frac{2}{a(a+2)}\rho(0) - \frac{1}{a(a+2)}\rho(-1) = 0,$$

and hence $\rho(1) = 2/(a^2 + 2a - 1)$. Solving the linear system

$$\begin{cases} A_1 + A_2 = 1 \\ A_1/a - A_2/(a+2) = 2/(a^2 + 2a - 1) \end{cases}$$

yields

$$A_1 = \frac{a(a+3)}{2(a^2+2a-1)}, \qquad A_2 = \frac{(a-1)(a+2)}{2(a^2+2a-1)}.$$

(c) The 1- and 2-steps ahead predictions are

$$X_{n+1|n} = \frac{1}{a(a+2)}(2X_n + X_{n-1}),$$

and

$$X_{n+2|n} = \frac{1}{a(a+2)}(2X_{n+1|n} + X_n) = \frac{2}{a^2(a+2)^2}(2X_n + X_{n-1}) + \frac{1}{a(a+2)}X_n.$$

4. Consider the ARMA(1,2) process $X_t + \alpha X_{t-1} = Z_t + \beta Z_{t-2}$.

(a) (10 points) When α and β are real numbers, under which conditions does the process X_t becomes stationary and invertible?

(b) (10 points) Given the observations $X_1, \ldots, X_n$, what is your best prediction for X_{n+1} and X_{n+2} at the forecast origin X_n? What is the corresponding forecasting error?

(c) (20 points) Compute its autocorrelation coefficient at lag 1, i.e., $\rho(1)$? (Hint: Compute its autocovariance function at lags 0 and 1, i.e., $\gamma(0)$ and $\gamma(1)$ first)

Solution: (a) The stationary condition for X_t is $|\alpha| < 1$. The invertibility condition for X_t is that the roots of $1 + \beta B^2 = 0$ lie outside the unit circle, that is, $-1 < \beta < 0$ or $0 < \beta < 1$.

(b) Given the observations $X_1, \ldots, X_n$, the 1- and 2-step ahead forecasts are

$$X_{n+1|n} = E(X_{n+1}|\mathcal{F}_n) = -\alpha X_n + \beta Z_{n-1},$$

$$X_{n+2|n} = E(X_{n+2}|\mathcal{F}_n) = -\alpha E(X_{n+1|n}|\mathcal{F}_n) + \beta Z_n = -\alpha(-\alpha X_n + \beta Z_{n-1}) + \beta Z_n,$$

and their corresponding prediction errors are

$$e_n(1) = X_{n+1} - X_{n+1|n} = Z_{n+1},$$

$$e_n(2) = X_{n+2} - X_{n+2|n} = Z_{n+2} - \alpha e_n(1) = Z_{n+2} - \alpha Z_{n+1}.$$

(c) We first notice that

$$\gamma(0) = \alpha^2\gamma(0) + (1+\beta^2)\sigma^2 - 2\alpha\beta \text{Cov}(X_{t-1}, Z_{t-2}),$$

$$\text{Cov}(X_{t-1}, Z_{t-2}) = \text{Cov}(-\alpha X_{t-2} + Z_{t-1} + \beta Z_{t-3}, Z_{t-2}) = -\alpha \text{Cov}(X_{t-2}, Z_{t-2}),$$

and

$$\text{Cov}(X_{t-2}, Z_{t-2}) = \text{Cov}(-\alpha X_{t-3} + Z_{t-2} + \beta Z_{t-4}, Z_{t-2}) = \sigma^2.$$

We then have $(1-\alpha^2)\gamma(0) = (1 + \beta^2 + 2\alpha^2\beta)\sigma^2$, or

$$\gamma(0) = \frac{1 + \beta^2 + 2\alpha^2\beta}{1-\alpha^2}.$$

We now calculate $\gamma(1)$. Note that

$$\begin{aligned}
\gamma(1) &= \text{Cov}(X_t, X_{t-1}) = -\alpha\gamma(0) + \beta \text{Cov}(Z_{t-2}, X_{t-1}) \\
&= -\alpha\gamma(0) + \beta \text{Cov}(Z_{t-2}, -\alpha X_{t-2} + Z_{t-1} + \beta Z_{t-3}) = -\alpha\gamma(0) - \alpha\beta \text{Cov}(Z_{t-2}, X_{t-2}) \\
&= -\alpha\gamma(0) - \alpha\beta\sigma^2
\end{aligned}$$

Therefore

$$\rho(1) = \gamma(1)/\gamma(0) = -\alpha - \frac{\alpha\beta}{\gamma(0)}\sigma^2 = -\alpha - \frac{\alpha\beta(1-\alpha^2)}{1+\beta^2+2\alpha^2\beta}.$$