AMS316 Homework #1

This problem set is used as a self-test of your background on AMS311 and AMS315, it doesn't represent the format or level of homework in AMS316. The grade of homework 1 will be not counted for final grade.

1. Let the random variable X and Y have joint distribution

   $$P(X = a, Y = 0) = P(X = 0, Y = a) = P(X = -a, Y = 0) = P(X = 0, Y = -a) = 1/4.$$

   Show that $X - Y$ and $X + Y$ are independent.

2. The beta function $B(a, b)$ is given by $B(a, b) = \int_0^1 v^{a-1}(1-v)^{b-1}dv$; $a > 0, b > 0$. The beta distribution has density

   $$f(x) = \frac{1}{B(a, b)} x^{a-1}(1-x)^{b-1}, \qquad \text{for } 0 < x < 1.$$

   If X has the beta distribution, show that $E(X) = B(a+1, b)/B(a, b)$. What is $\mathrm{Var}(X)$?

3. A molecule M has velocity $w = (x, y, z)$ in Cartesian coordinates. Suppose that x, y and z have joint density:

   $$f(x, y, z) = (2\pi\sigma^2)^{-3/2} \exp\Big(-\frac{1}{2\sigma^2}(x^2 + y^2 + z^2)\Big).$$

   Show that the density of the magnitude $|w| = \sqrt{x^2 + y^2 + z^2}$ of w is

   $$f(w) = \Big(\frac{2}{\pi}\Big)^{1/2} \sigma^{-3} w^2 \exp\Big(-\frac{w^2}{2\sigma^2}\Big), \qquad w > 0.$$

   (Hint: Argue that x, y, z are independent first, and then notice that w^2/σ^2 follows a χ^2 distribution).

4. Consider a continuous random variable X with density function $f(x)$, mean μ and variance σ^2. Show that

   $$\sigma^2 \geq \int_{|x-\mu|\geq\epsilon} (x-\mu)^2 f(x)dx \geq \epsilon^2 \int_{|x-\mu|\geq\epsilon} f(x)dx,$$

   and furthermore,

   $$P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}.$$

5. Let $X_1, \ldots, X_n$ are independent and identically distributed random variables with mean μ and variance σ^2. Apply the result in Problem 4 for $X = \sum_{i=1}^n X_i/n$ to show the law of large number.