AMS 316 HW#1 Solution.

1. From the joint distribution, we know

$X+Y, X-Y$:	(a, a)	$(a, -a)$	$(-a, a)$	$(-a, -a)$
Prob :	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

$X+Y$:	a	$-a$
Prob :	$\frac{1}{2}$	$\frac{1}{2}$

$X-Y$:	a	$-a$
Prob :	$\frac{1}{2}$	$\frac{1}{2}$

$\Rightarrow \quad P(X+Y, X-Y) = P(X+Y) \cdot P(X-Y)$

$\Rightarrow \quad X+Y \perp X-Y$

2.
$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \frac{1}{B(a,b)} \int_0^1 x^a (1-x)^{b-1} = \frac{B(a+1, b)}{B(a,b)}$$

$Var(X) = E(X^2) - (E(X))^2$ and $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \frac{1}{B(a,b)} \int_0^1 x^{a+1} (1-x)$

$$= \frac{B(a+2, b)}{B(a,b)}$$

$$\Rightarrow Var(X) = \frac{B(a+2, b)}{B(a,b)} - \frac{B^2(a+1, b)}{B^2(a,b)}$$

3. From x, y, z joint density, we can get x, y and z are independent

And x, y & $z \overset{iid}{\sim} N(0, \sigma^2)$

$\Rightarrow \quad \left(\frac{x^2}{\sigma^2} + \frac{y^2}{\sigma^2} + \frac{z^2}{\sigma^2}\right) \sim \chi^2(3)$, Make $a = \frac{1}{\sigma^2}(x^2+y^2+z^2)$.

We know $a \sim \chi^2(3)$ and $f_a(a) = (2\pi)^{-\frac{1}{2}} \cdot a^{\frac{1}{2}} \cdot e^{-\frac{a}{2}}$ $\quad 0 \leq a < \infty$

$\Rightarrow \quad w^2 = \sigma^2 a$, $a = \frac{w^2}{\sigma^2}$ $\quad \frac{\partial a}{\partial w} = \frac{2w}{\sigma^2}$

$$\Rightarrow f(w) = (2\pi)^{-\frac{1}{2}} \frac{w}{\sigma} \cdot e^{-\frac{w^2}{2\sigma^2}} \cdot \frac{2w}{\sigma^2}$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sigma^{-3} w^2 \exp\left(-\frac{w^2}{2\sigma^2}\right), \quad w > 0$$

4. $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$ according to the definition of Variance.

and $f(x) \geq 0$, $(x-\mu)^2 \geq 0$, so $\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \geq \int_{|x-\mu|\geq\varepsilon} (x-\mu)^2 f(x)\,dx$

$$\Rightarrow \sigma^2 \geq \int_{|x-\mu|\geq\varepsilon} (x-\mu)^2 f(x)\,dx$$

When $|x-\mu| \geq \varepsilon$, $(x-\mu)^2 \geq \varepsilon^2$, so $\int_{|x-\mu|\geq\varepsilon} (x-\mu)^2 f(x)\,dx \geq \int_{|x-\mu|\geq\varepsilon} \varepsilon^2 f(x)\,dx$

$$\Rightarrow \int_{|x-\mu|\geq\varepsilon} (x-\mu)^2 f(x)\,dx \geq \varepsilon^2 \int_{|x-\mu|\geq\varepsilon} f(x)\,dx$$

$$\Rightarrow \varepsilon^2 \int_{|x-\mu|\geq\varepsilon} f(x)\,dx \leq \sigma^2$$

$$\int_{|x-\mu|\geq\varepsilon} f(x)\,dx \leq \frac{\sigma^2}{\varepsilon^2}$$

$$P(|X-\mu|\geq\varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

5. $X = \sum_{i=1}^{n} X_i/n$ and $X_1, \cdots, X_n \overset{iid.}{\sim} (\mu, \sigma^2)$

$$\Rightarrow X \sim (\mu, \frac{\sigma^2}{n})$$

Using the last formula in Q4, we have

$$P(|X-\mu|\geq\varepsilon) \leq \frac{\sigma^2}{n\cdot\varepsilon^2}$$

$$\Rightarrow \lim_{n\to\infty} P(|X-\mu|\geq\varepsilon) = 0$$

$$\Rightarrow \lim_{n\to\infty} P(|X-\mu|<\varepsilon) = 1$$, LLN is showed.