AMS316 Homework 3 Solution

3.1 Solution:

$$\begin{aligned}\gamma(k) &= cov(X_t, X_{t+k}) \\ &= cov(Z_t + 0.7Z_{t-1} - 0.2Z_{t-2}, Z_{t+k} + 0.7Z_{t+k-1} - 0.2Z_{t+k-2})\end{aligned}$$

when $k = 0$,

$$\begin{aligned}\gamma(0) &= Var(X_t) = Var(Z_t) + 0.7Var(Z_t) - 0.2Var(Z_t) \\ &= 1.5Var(Z_t);\end{aligned}$$

when $k = 1$,

$$\begin{aligned}\gamma(1) &= cov(Z_t + 0.7Z_{t-1} - 0.2Zt - 2, Z_{t-1} + 0.7Z_t - 0.2Z_{t-1}) \\ &= 0.7Var(Z_t) - 0.14Var(Z_t) = 0.56Var(Z_t)\end{aligned}$$

Hence $\rho(1) = \frac{\gamma(1)}{\gamma(0)} = 0.37$. When $k = 2$,

$$\begin{aligned}\gamma(2) &= cov(Z_t + 0.7Z_{t-1} - 0.2Z_{t-2}, Z_{t+2} + 0.7Z_{t+1} - 0.2Z_t) \\ &= -0.2cov(Z_t, Z_t) \\ &= -0.2Var(Z_t)\end{aligned}$$

Hence $\rho(2) = \frac{\gamma(2)}{\gamma(0)} = \frac{-0.2}{1.5} = -0.13$. It's easy to find out ac.f when k $=$ $-1, -2$, therefore,

$$\rho(k) = \begin{cases} 1 & : \quad k = 0 \\ 0.37 & : \quad k = \pm 1 \\ -0.13 & : \quad k = \pm 2 \\ 0 & : \quad otherwise \end{cases} \tag{1}$$

3.2 Solution: For $X_t = \sum_{k=0}^{m} \frac{Z_{t-k}}{m+1}$, we have

$$E(X_t) = E\left(\sum_{k=0}^{m} \frac{Z_{t-k}}{m+1}\right) = 0. \tag{2}$$

Since $\rho(k) = \frac{\gamma(k)}{\gamma(0)} = corr(X_t, X_{t+k})$,

$$\begin{aligned}\gamma(0) &= Var(X_t) = Var(\sum_{k=0}^{m} \frac{Z_{t-k}}{m+1}) \\ &= \frac{1}{(m+1)^2} \sum_{k=0}^{m} var(Z_{t-k}) = \sigma^2 \frac{1}{m+1},\end{aligned}$$

when $m \geq k \geq 1$,

$$\begin{aligned}\gamma(k) &= cov(X_t, X_{t+k}) = cov\left(\sum_{k=0}^{m} \frac{Z_{t-k}}{m+1}, \sum_{k=0}^{m} \frac{Z_{t-k+i}}{m+1}\right) \\ &= E\left[(\sum_{t=0}^{m-k} \frac{Z_{t-k}}{m+1})(\sum_{i=0}^{m-k} \frac{Z_{t-k+i}}{m+1})\right] = \sigma^2 \left(\sum_{i=0}^{m-k} \frac{1}{(m+1)^2}\right) \\ &= \sigma^2 \frac{m-k+1}{(m+1)^2}\end{aligned}$$

when $k \geq m$, $\gamma(k) = 0$. Hence

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)} = \begin{cases} \frac{\sigma^2(m-k+1)}{(m+1)^2} / \frac{\sigma^2(m+1)}{(m+1)^2} & : \quad k = 0, 1, ..., m \\ 0 & : \quad k \geq m \end{cases} \tag{3}$$

Simplifying it yields

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)} = \begin{cases} \frac{m-k+1}{m+1} & : \quad k = 0, 1, ..., m \\ 0 & : \quad k \geq m \end{cases} \tag{4}$$

3.3 Solution: For process $X_t = Z_t + c(Z_{t-1} + Z_{t-2} + ...)$, we have

$$\begin{aligned}E(X_t) &= E(Z_t + cZ_{t-1} + cZ_{t-2} + ...) \\ &= E(Z_t) + cE(Z_{t-1}) + cE(Z_{t-2}) + ... = 0\end{aligned}$$

and

$$var(X_t) = var(Z_t) + var(C(Z_{t-1} + Z_{t-2} + ...) = \infty$$

Since variance is infinite, therefore, X_t is non-stationary. For $Y_t = X_t - X_{t-1}$, we have

$$\begin{aligned} E(Y_t) &= E(X_t - X_{t-1}) \\ &= E\left(Z_t - Z_{t-1} + cZ_{t-1} - cZ_{t-1} + cZ_{t-2} - cZ_{t-3} + ...\right) \\ &= E(Z_t - Z_{t-1} + cZ_{t-1}) = E(Z_t) - E(Z_{t-1}) + cE(Z_{t-1}) \\ &= 0 - 0 + c * 0 = 0 \end{aligned}$$

$$\begin{aligned} \gamma(k) &= cov(Y_t, Y_{t+k}) = cov(X_t - X_{t-1}, X_{t+k} - X_{t+k-1}) \\ &= cov[Z_t + c(Z_{t-1} + Z_{t-2}) + ... - Z_{t-1} - c(Z_{t-2} + Z_{t-3} + ...) \\ &+ Z_{t+k} + c(Z_{t+k-1} + ...) - Z_{t+k-1} - c(Z_{t+k-2} + Z_{t+k-3} + ...)] \\ &= cov(Z_t + (c-1)Z_{t-1}, Z_{t+k} + (c-1)Z_{t+k-1}) \\ &= E[(Z_t + (c-1)Z_{t-1})(Z_{t+k} + (c-1)Z_{t+k-1})] \end{aligned}$$

$$\gamma(k) = \begin{cases} cov(Y_t, Y_t) = Var(Y_t) = ((c-1)^2 + 1)\sigma^2 & : \quad k = 0 \\ cov(Y_t, Y_{t+1}) = (c-1)\sigma^2 & : \quad k = \pm 1 \\ cov(Y_t, Y_{t+k}) = 0 & : \quad otherwise \end{cases}$$

Hence Y_t is stationary and

$$\rho(k) = \begin{cases} \gamma(0)/\gamma(0) = 1 & : \quad k = 0 \\ \gamma(k)/\gamma(0) = \frac{c-1}{1+(c-1)^2} & : \quad k = \pm 1 \\ \gamma(k)/\gamma(0) = 0 & : \quad otherwise \end{cases} \tag{5}$$

3.5 Solution: For process $X_t = \mu + Z_t + \beta Z_{t-1}$, we have that, when $k = 0$,

$$\gamma(0) = Var(X_t) = Var(Z_t) + \beta^2 Var(Z_t) = (1 + \beta^2)Var(Z_t) \tag{6}$$

when $k = 1$,

$$\begin{aligned} \gamma(1) &= cov(\mu + Z_t + \beta Z_{t-1}, \mu + Z_{t+1} + \beta Z_t) \\ &= \beta Var(Z_t) \end{aligned}$$

when $k = 2$,

$$\begin{aligned}\gamma(2) &= cov(\mu + Z_t + \beta Z_{t-1}, \mu + Z_{t-1} + \beta Z_{t-2}) \\ &= 0\end{aligned}$$

when $k = -1$

$$\begin{aligned}\gamma(-1) &= cov(\mu + Z_t + \beta Z_{t-1}, \mu + Z_{t-1} + \beta Z_{t-2}) \\ &= \beta Var(Z_t)\end{aligned}$$

$$\rho(k) = \begin{cases} 1 & : \quad k = 0 \\ \frac{\beta}{(1+\beta^2)} & : \quad k = \pm 1 \\ 0 & : \quad otherwise \end{cases} \tag{7}$$

Hence the ac.f does not depend on μ.