AMS316 Homework 4 Solution

1. Since $|\alpha| < 1$, we have

$$X_t = \frac{Z_t}{(1 - \alpha ß)} = Z_t + \alpha Z_{t-1} + \alpha^2 Z_{t-2} + ...$$

and

$$Var(X_t) = \frac{\sigma_Z^2}{(1 - \alpha^2)}$$

Furthermore,

$$\gamma(k) = E[X_t X_{t+k}] = E[\sum \alpha^i Z_{t-i} \sum \alpha^j Z_{t+k-j}]$$
$$= \sigma_Z^2 \sum_{i=0}^{\infty} \alpha^i \alpha^(k + i) = \alpha^k \frac{\sigma_Z^2}{(1 - \alpha^2)}$$

Hence

$$\rho(k) = \alpha^{|k|} \qquad k = 0, \pm 1, \pm 2, ...$$

and $\rho(k) = 0.7^{|k|}$.

2. The roots of equation $y^2 - \frac{1}{3}y - \frac{2}{9} = 0$ are 2/3 and -1/3, then the ACFs are given by

$$\rho(k) = A_1 \big(\frac{2}{3}\big)^{|k|} + A_2 \big(-\frac{1}{3}\big)^{|k|}.$$

Since $A_1 + A_2 = \rho(0) = 1$ and

$$\rho(1) = \frac{2}{3} A_1 - \frac{1}{3} A_2$$

and $\rho(1) = 3/7$, solving for A_1 and A_2 gives $A_1 = \frac{16}{21}$ and $A_2 = \frac{5}{21}$.

3. The roots of equation $y^2 - \frac{1}{12}y - \frac{1}{12} = 0$ are 1/3 and $-1/4$. Then the ACFs are given by

$$\rho(k) = A_1 \big(\frac{1}{3}\big)^{|k|} + A_2 \big(-\frac{1}{4}\big)^{|k|}.$$

Since $A_1 + A_2 = \rho(0) = 1$ and

$$\rho(1) = \frac{1}{3}A_1 - \frac{1}{4}A_2$$

and $\rho(1) = 1/11$, solving for A_1 and A_2 gives $A_1 = \frac{45}{77}$ and $A_2 = \frac{32}{77}$.

4. (a) The root of $1 - 0.3B = 0$ is $1/0.3$, which lies outside the unit circle, so X_t is stationary. AR process is always invertible.

$$X_t = (\frac{1}{1-0.3B}Z_t) = (1 + 0.3B + 0.3^2B^2 + ...)Z_t$$

(b) The roots of $1 - 1.3B + 0.4B^2 = 0$ are 2 and 1.25. They all lie outside of unit circle, so it's invertible. MA process is always stationary.

$$X_t = (1 - 1.3B + 0.4B^2)Z_t.$$

(c) The root of $1 - 0.5B = 0$ is 2, which lies outside the unit circle, so X_t is stationary. The roots of $1 - 1.3B + 0.4B^2 = 0$ are 2 and 1.25, which lie outside the unit circle, so X_t is invertible.

$$(1 - 0.5B)X_t = (1 - 1.3B + 0.4B^2)Z_t.$$