1.

$$\gamma(0) = Var(X_t) = Var(\alpha X_{t-1} + Z_t + \beta Z_{t-1})$$

$$= \alpha^2 Var(X_{t-1}) + \sigma^2 + \beta^2\sigma^2 + 2Cov(\alpha X_{t-1}, \beta Z_{t-1})$$

$$= \alpha^2 Var(X_t) + (1+\beta^2)\sigma^2 + 2\alpha\beta\sigma^2$$

$$\Rightarrow (1-\alpha^2) Var(X_t) = (1+\beta^2+2\alpha\beta)\sigma^2$$

$$Var(X_t) = \left(\frac{1+\beta^2+2\alpha\beta}{1-\alpha^2}\right)\sigma^2 = \gamma(0)$$

$$\gamma(1) = Cov(X_t, X_{t-1}) = Cov(\alpha X_{t-1} + Z_t + \beta Z_{t-1}, X_{t-1})$$

$$= \alpha Var(X_{t-1}) + \beta Cov(Z_{t-1}, X_{t-1})$$

$$= \alpha\gamma(0) + \beta\sigma^2$$

$$= \frac{(1+\alpha\beta)(\alpha+\beta)}{1-\alpha^2}\sigma^2$$

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{(1+\alpha\beta)(\alpha+\beta)}{1+\beta^2+2\alpha\beta}$$

$$\gamma(k) = Cov(X_{t+k}, X_k) = Cov(\alpha X_{t+k-1} + Z_{t+k} + \beta Z_{t+k-1}, X_t)$$

$$= \alpha \underbrace{Cov(X_{t+k-1}, X_t)}_{=\gamma(k-1)} + \underbrace{Cov(Z_{t+k} + \beta Z_{t+k-1}, X_t)}_{=0}$$

$$\Rightarrow \gamma(k) = \alpha\gamma(k-1) \qquad k \geq 2$$

$$\Rightarrow \rho(k) = \alpha\rho(k-1) \qquad k \geq 2$$

2.

(a). Compare with ARIMA model $\phi(B)(1-B)^d X_t = \theta(B) Z_t$

we find out $p = d = q = 1$

(b) The roots of $(1-B)(1-0.2B)=0$ are $B=1$ or $B=2$ and $B=1$ lies on the unit circle, so the process is non-stationary.

The roots of $1-0.5B=0$ is $B=2$. So the process is invertible.

(c). $X_t = (1-0.5B)(1-B)^{-1}(1-0.2B)^{-1}Z_t$

$$=(1-0.5B)(1+B+B^2+\cdots)(1+0.2B+0.2^2B^2+\cdots)Z_t$$

$$=(1+0.7B+0.64B^2+0.628B^3+\cdots)Z_t$$

The first three weights are 0.7, 0.64, 0.628. Since they are decreasing slowly, the process is non-stationary.

(d) $Z_t = (1-B)(1-0.2B)(1-0.5B)^{-1}X_t$

$$=(1-1.2B+0.2B^2)(1+0.5B+0.5^2B^2+\cdots)X_t$$

$$=(1-0.7B-0.15B^2-0.075B^3+\cdots)X_t$$

The first three weights are $-0.7, -0.15, -0.075$. Since they are decreasing quickly, the process is invertible.

3.

$(1-B-cB^2)X_t = Z_t$ is stationary.

$\Rightarrow$ $1-B-cB^2=0$ and both roots lies outside unit circle.

$\Rightarrow$ roots of $y^2-y-c=0$ lies inside unit circle.

$$\Rightarrow \quad \left| \frac{1 \pm \sqrt{1+4c}}{2} \right| < 1$$

(i) when $1+4c \geq 0$ which $c \geq -\frac{1}{4}$

$$-1 \leq \frac{1-\sqrt{1+4c}}{2} \leq \frac{1+\sqrt{1+4c}}{2} < 1$$

$$\Rightarrow \quad -\frac{1}{4} \leq c < 0$$

(ii) when $1+4c < 0$ which $c < -\frac{1}{4}$

$$\Rightarrow \quad \left\| \frac{1 \pm \sqrt{-1-4c}\, i}{2} \right\| < 1$$

$$\Rightarrow \quad \frac{1}{4} + \frac{-1-4c}{4} < 1 \quad \Rightarrow \quad c > -1$$

$$\Rightarrow \quad -1 < c < -\frac{1}{4}$$

Combine with (i), we have the stationary condition

$$-1 < c < 0$$

If $c = -\frac{3}{16}$.

$$X_t - X_{t-1} + \frac{3}{16} X_{t-2} = Z_t$$

roots of $y^2 - y + \frac{3}{16} = 0$ are $\pi_1 = \frac{3}{4}$ and $\pi_2 = \frac{1}{4}$

Yule - Walker Equation:

$$\rho(0) = A_1 + A_2 = 1$$

$$\rho(1) = \frac{3}{4} A_1 + \frac{1}{4} A_2 = \frac{16}{19}$$

$$\Rightarrow A_1 = \frac{45}{38}, \quad A_2 = \frac{-7}{38}$$

$$\Rightarrow \rho(k) = \frac{45}{38} \left(\frac{3}{4}\right)^{|k|} - \frac{7}{38} \left(\frac{1}{4}\right)^{|k|}$$

4.

$1 - B - cB^2 + cB^3 = 0$ has one root at $B=1$, and it doesn't lie outside unit circle. So the process is non-stationary for any value of c.