AMS 316 Final Solution

December 15, 2010

1. $x_t = a + bt + ct^2 + s_t + y_t$

   (a) trend: $a + bt + ct^2$, seasonal: s_t

   (b) $s_t \equiv 0$, $(1-B)^2 x_t = (1-2B+B^2)x_t = a + bt + ct^2 + y_t - 2a - 2b(t-1) - 2c(t-1)^2 + a + b(t-2) + c(t-2)^2 = 2c + y_t - 2y_{t-1} + y_{t-2}$. Since y_t is stationary, $(1-B)^2 x_t$ is stationary.

2. $x_t = z_t + \theta_1 z_{t-1} + \theta_2 z_{t-2}$

   (a) ACVF:

$$
\begin{aligned}
\gamma_x(k) &= cov(z_t + \theta_1 z_{t-1} + \theta_2 z_{t-2}, z_{t+k} + \theta_1 z_{t+k-1} + \theta_2 z_{t+k-2}) \\
&= \begin{cases} 0 & k > 2 \\ \sigma^2 \cdot \theta_2 & k = 2 \\ \sigma^2 \cdot \theta_1(1+\theta_2) & k = 1 \\ \sigma^2 \cdot (1 + \theta_1^2 + \theta_2^2) & k = 0 \\ \gamma_x(-k) & k < 0 \end{cases}
\end{aligned}
$$

ACF:

$$\begin{aligned}
\rho_x(k) &= \gamma_x(k)/\gamma_x(0) \\
&= \begin{cases} 0 & k > 2 \\ \theta_2/(1+\theta_1^2+\theta_2^2) & k = 2 \\ \theta_1(1+\theta_2)/(1+\theta_1^2+\theta_2^2) & k = 1 \\ 1 & k = 0 \\ \rho_x(-k) & k < 0 \end{cases}
\end{aligned}$$

(b) $y_t = \alpha x_{t-1} + x_t$. ACVF of y_t:

$$\begin{aligned}
\gamma_y(k) &= cov(y_t, y_{t+k}) = cov(\alpha x_{t-1} + x_t, \alpha x_{t+k-1} + x_{t+k}) \\
&= (1+\alpha^2)\gamma_x(k) + \alpha\gamma_x(k-1) + \alpha\gamma_x(k+1) \\
&= \begin{cases} 0 & k > 3 \\ \sigma^2 \cdot \alpha\theta_2 & k = 3 \\ \sigma^2 \left[(1+\alpha^2)\theta_2 + \alpha\theta_1(1+\theta_2)\right] & k = 2 \\ \sigma^2 \left[(1+\alpha^2)\theta_1(1+\theta_2) + \alpha(1+\theta_1^2+\theta_2^2) + \alpha\theta_2\right] & k = 1 \\ \sigma^2 \left[2\alpha\theta_1(1+\theta_2) + (1+\alpha^2)(1+\theta_1^2+\theta_2^2)\right] & k = 0 \\ \gamma_y(-k) & k < 0 \end{cases}
\end{aligned}$$

ACF:

$$
\begin{aligned}
\rho_y(k) &= \gamma_y(k)/\gamma_y(0) \\
&= \begin{cases}
0 & k > 3 \\
\frac{\alpha\theta_2}{2\alpha\theta_1(1+\theta_2)+(1+\alpha^2)(1+\theta_1^2+\theta_2^2)} & k = 3 \\
\frac{(1+\alpha^2)\theta_2+\alpha\theta_1(1+\theta_2)}{2\alpha\theta_1(1+\theta_2)+(1+\alpha^2)(1+\theta_1^2+\theta_2^2)} & k = 2 \\
\frac{(1+\alpha^2)\theta_1(1+\theta_2)+\alpha(1+\theta_1^2+\theta_2^2)+\alpha\theta_2}{2\alpha\theta_1(1+\theta_2)+(1+\alpha^2)(1+\theta_1^2+\theta_2^2)} & k = 1 \\
1 & k = 0 \\
\rho_y(-k) & k < 0
\end{cases}
\end{aligned}
$$

3. $x_t = \frac{1}{a(a+1)}x_{t-1} + \frac{1}{a(a+1)}x_{t-2} + z_t$.

   (a) $z_t = \left(1 - \frac{1}{a(a+1)}B - \frac{1}{a(a+1)}B^2\right)x_t$, set $Y = \frac{1}{B}$, we get $Y^2 - \frac{1}{a(a+1)}Y - \frac{1}{a(a+1)} = 0$. There are two roots: $Y_1 = \frac{1}{a}$, $Y_2 = -\frac{1}{a+1}$. The stationarity condition: $|Y_i| < 1 \Rightarrow a < -2$ or $a > 1$.

   (b) $\rho(k) = A_1 Y_1^{|k|} + A_2 Y_2^{|k|}$. So $\rho(0) = A_1 + A_2 = 1$, $\rho(1) = \frac{1}{a}A_1 - \frac{1}{a+1}A_2 = \frac{1}{a^2+a-1}$. So the coefficients are: $A_1 = \frac{a^3+2a^2}{2a^3+3a^2-a-1}$, $A_2 = \frac{a^3+a^2-a-1}{2a^3+3a^2-a-1}$. So $\rho(k) = \frac{a^3+2a^2}{2a^3+3a^2-a-1}(\frac{1}{a})^{|k|} + \frac{a^3+a^2-a-1}{2a^3+3a^2-a-1}(-\frac{1}{a+1})^{|k|}$.

   (c) $x_{n+1} = \frac{1}{a(a+1)}x_n + \frac{1}{a(a+1)}x_{n-1} + z_{n+1}$,

   then $x_n(1) = E(x_{n+1}|x_1,\ldots,x_n) = \frac{1}{a(a+1)}x_n + \frac{1}{a(a+1)}x_{n-1}$,

   $e_n(1) = x_{n+1} - x_n(1) = z_{n+1}$, $var(e_n(1)) = \sigma^2$.

   $x_{n+2} = \frac{1}{a(a+1)}x_{n+1} + \frac{1}{a(a+1)}x_n + z_{n+2}$,

   then $x_n(2) = E(x_{n+2}|x_1,\ldots,x_n) = \frac{1}{a(a+1)}x_n(1) + \frac{1}{a(a+1)}x_n = \frac{1}{a(a+1)}\left[\frac{1}{a(a+1)}x_n + \frac{1}{a(a+1)}x_{n-1}\right] + \frac{1}{a(a+1)}x_n$,

   $e_n(2) = x_{n+2} - x_n(2) = z_{n+2} + \frac{1}{a(a+1)}z_{n+1}$, $var(e_n(2)) = \left[1 + \left(\frac{1}{a(a+1)}\right)^2\right]\sigma^2$.

4. $x_t + \phi x_{t-1} = z_t - \theta z_{t-1}$.

   (a) stationary: $|\phi| < 1$; invertible: $|\theta| < 1$.

(b) $x_t(1+\phi B) = z_t(1-\theta B) \Rightarrow x_t = \frac{1-\theta B}{1+\phi B} z_t$. Define $\Psi(B) = \frac{1-\theta B}{1+\phi B} = 1 + \psi_1 B + \psi_2 B^2 + \cdots$, where $\psi_k = (-1)^k \phi^{k-1}(\phi+\theta)$.

So $x_t = z_t + \psi_1 z_{t-1} + \psi_2 z_{t-2} + \cdots$.

$E\left(x_{t-k}x_t + \phi x_{t-k}x_{t-1}\right) = E\left[x_{t-k}\left(z_t - \theta z_{t-1}\right)\right]$

$\gamma(k)+\phi\gamma(k-1) = E\left[\left(z_{t-k} + \psi_1 z_{t-k-1} + \psi_2 z_{t-k-2} + \cdots\right)\left(z_t - \theta z_{t-1}\right)\right]$

For $k=0$, $\gamma(0) + \phi\gamma(-1) = \sigma^2\left[1 + \theta(\phi+\theta)\right]$;

For $k=1$, $\gamma(1) + \phi\gamma(0) = -\theta\sigma^2$;

For $k>1$, $\gamma(k) = -\phi\gamma(k-1)$.

So $\gamma(0) = \frac{1+\theta^2+2\phi\theta}{1-\phi^2}\sigma^2$, $\gamma(1) = -\frac{(\phi+\theta)(1+\phi\theta)}{1-\phi^2}\sigma^2$, $\gamma(2) = \frac{\phi(\phi+\theta)(1+\phi\theta)}{1-\phi^2}\sigma^2$.

$\rho(1) = -\frac{(\phi+\theta)(1+\phi\theta)}{1+\theta^2+2\phi\theta}$, $\rho(2) = \frac{\phi(\phi+\theta)(1+\phi\theta)}{1+\theta^2+2\phi\theta}$.

(c) $x_{n+1} = -\phi x_n + z_{n+1} - \theta z_n$,

then $x_n(1) = E(x_{n+1}|x_1, \ldots, x_n) = -\phi x_n - \theta z_n$,

$e_n(1) = x_{n+1} - x_n(1) = z_{n+1}$, $var(e_n(1)) = \sigma^2$.

$x_{n+2} = -\phi x_{n+1} + z_{n+2} - \theta z_{n+1}$,

then $x_n(2) = E(x_{n+2}|x_1, \ldots, x_n) = -\phi x_n(1) = \phi^2 x_n + \phi\theta z_n$,

$e_n(2) = x_{n+2} - x_n(2) = z_{n+2} - (\phi+\theta) z_{n+1}$, $var(e_n(2)) = \left[1 + (\phi+\theta)^2\right]\sigma^2$.