r_p = the expected return corresponding to the portfolio ω_p.

r_q = the expected return corresponding to the portfolio ω_q

$$Cov(r_p, r_q) = Cov(\omega_p^T r, \omega_q^T r) = Cov(r^T\omega_p, \omega_q^T r)$$

$$\omega_p = \frac{1}{D}\{B\Sigma^{-1}\mathbb{1} - A\Sigma^{-1}\mu + \mu_p(C\Sigma^{-1}\mu - A\Sigma^{-1}\mathbb{1})\}$$

$$= \frac{1}{D}\{(B - A\mu_p)\Sigma^{-1}\mathbb{1} + (C\mu_p - A)\Sigma^{-1}\mu\}$$

$$\omega_q = \frac{1}{D}\{(B - A\mu_q)\Sigma^{-1}\mathbb{1} + (C\mu_q - A)\Sigma^{-1}\mu\}$$

$$Cov(r^T\Sigma^{-1}\mathbb{1}, \mathbb{1}^T\Sigma^{-1}r) = \mathbb{1}^T\Sigma^{-1}Cov(r)\Sigma^{-1}\mathbb{1} = \mathbb{1}^T\Sigma^{-1}\mathbb{1} = C$$

$$Cov(r^T\Sigma^{-1}\mu, \mu^T\Sigma^{-1}r) = \mu^T\Sigma^{-1}Cov(r)\Sigma^{-1}\mu = \mu^T\Sigma^{-1}\mu = B$$

$$Cov(r^T\Sigma^{-1}\mathbb{1}, \mu^T\Sigma^{-1}r) = \mu^T\Sigma^{-1}Cov(r)\Sigma^{-1}\mathbb{1} = \mu^T\Sigma^{-1}\mathbb{1} = A = \mathbb{1}^T\Sigma^{-1}\mu = Cov(r^T\Sigma^{-1}\mu, \mathbb{1}^T\Sigma^{-1}r)$$

So, $Cov(r_p, r_q) = Cov(r^T\omega_p, \omega_q^T r) = \frac{1}{D^2}\{(B - A\mu_p)(B - A\mu_q)C +$

$$(B - A\mu_p)(C\mu_q - A)A + (C\mu_p - A)(B - A\mu_q)A + (C\mu_p - A)(C\mu_q - A)B\}$$

$$= \frac{1}{D^2}\{B^2C - ABC\mu_q - ABC\mu_p + \cancel{A^2C\mu_p\mu_q} \cancel{+ABC\mu_q} - A^2B - A^2C\mu_p\mu_q + A^3\mu_p$$

$$\cancel{+ABC\mu_p} - \cancel{A^2C\mu_p\mu_q} - \cancel{A^2B} + A^3\mu_q + BC^2\mu_p\mu_q - \cancel{ABC\mu_p} - ABC\mu_q + \cancel{A^2B}\}$$

$$= \frac{1}{D^2}\{(B^2C - A^2B) + (BC^2 - A^2C)\mu_p\mu_q + (A^3 - ABC)(\mu_p + \mu_q)\}$$

Since $D = BC - A^2$

$$= \frac{1}{D}\{B + C\mu_p\mu_q - A(\mu_p + \mu_q)\}$$

we know $\frac{C}{D}(\mu_p - \frac{A}{C})(\mu_q - \frac{A}{C}) + \frac{1}{C} = \frac{C}{D}(\mu_p\mu_q - \frac{A}{C}(\mu_p + \mu_q) + \frac{A^2}{C^2}) + \frac{1}{C}$

$$= \frac{1}{D}(C\mu_p\mu_q - A(\mu_p + \mu_q) + \frac{A^2}{C} + \frac{D}{C})$$

$$= \frac{1}{D}(B + C\mu_p\mu_q - A(\mu_p + \mu_q))$$

2. $\min_{w} W^T \Sigma W$ s.t. $W^T\mu + (1-W^T\mathbb{1})r_f = \mu_*$

the lagrangian function: $L = W^T\Sigma W - 2\lambda(W^T\mu + (1-W^T\mathbb{1})r_f - \mu_*)$

$$\begin{cases} \frac{\partial L}{\partial W} = 0 & \Sigma W - \lambda(\mu - r_f\mathbb{1}) = 0 \Rightarrow W = \lambda\Sigma^{-1}(\mu - r_f\mathbb{1}) \quad \text{— ①} \\ \Rightarrow & W^T = \lambda(\mu - r_f\mathbb{1})^T\Sigma^{-1} \quad \text{— ②} \\ \frac{\partial L}{\partial \lambda} = 0 & W^T\mu + (1-W^T\mathbb{1})r_f = \mu_* \quad \text{— ③} \end{cases}$$

plug ② into ③, we get,

$$\lambda(\mu - r_f\mathbb{1})^T\Sigma^{-1}\mu + (1 - \lambda(\mu - r_f\mathbb{1})^T\Sigma^{-1}\mathbb{1})r_f = \mu_*$$

$$\lambda(\mu - r_f\mathbb{1})^T\Sigma^{-1}\mu - \lambda(\mu - r_f\mathbb{1})^T\Sigma^{-1}\mathbb{1}r_f = \mu_* - r_f$$

$$\lambda(\mu - r_f\mathbb{1})^T\Sigma^{-1}(\mu - r_f\mathbb{1}) = \mu_* - r_f$$

$$\lambda = \frac{\mu_* - r_f}{(\mu - r_f\mathbb{1})^T\Sigma^{-1}(\mu - r_f\mathbb{1})} \quad \text{— ④}$$

plug ④ into ①, we get,

$$W_{eff} = \frac{\mu_* - r_f}{(\mu - r_f\mathbb{1})^T\Sigma^{-1}(\mu - r_f\mathbb{1})} \cdot \Sigma^{-1}(\mu - r_f\mathbb{1})$$

(1) for any $\alpha \in \mathbb{R}$, the portfolio $\alpha w_p + (1-\alpha) w_q$ is efficient

we let μ_p: the expected return of the portfolio w_p

μ_q: the expected return of the portfolio w_q.

we know from question 1, that,

$$w_{eff} = \frac{1}{D}\{(B\Sigma^{-1}\mathbb{1} - A\Sigma^{-1}\mu) + \mu_*(C\Sigma^{-1}\mu - A\Sigma^{-1}\mathbb{1})\}$$

So, $$w_p = \frac{1}{D}\{(B\Sigma^{-1}\mathbb{1} - A\Sigma^{-1}\mu) + \mu_p(C\Sigma^{-1}\mu - A\Sigma^{-1}\mathbb{1})\}$$

$$w_q = \frac{1}{D}\{(B\Sigma^{-1}\mathbb{1} - A\Sigma^{-1}\mu) + \mu_q(C\Sigma^{-1}\mu - A\Sigma^{-1}\mathbb{1})\}$$

$\mu_* = \alpha\mu_p + (1-\alpha)\mu_q$,

$$w_{eff} = \frac{1}{D}\{(B\Sigma^{-1}\mathbb{1} - A\Sigma^{-1}\mu) + (\alpha\mu_p + (1-\alpha)\mu_q)(C\Sigma^{-1}\mu - A\Sigma^{-1}\mathbb{1})\}$$

$$= \frac{1}{D}\{(\alpha + (1-\alpha))(B\Sigma^{-1}\mathbb{1} - A\Sigma^{-1}\mu) + (\alpha\mu_p + (1-\alpha)\mu_q)(C\Sigma^{-1}\mu - A\Sigma^{-1}\mathbb{1}$$

$$= \alpha \cdot \frac{1}{D}\{(B\Sigma^{-1}\mathbb{1} - A\Sigma^{-1}\mu) + \mu_p(C\Sigma^{-1}\mu - A\Sigma^{-1}\mathbb{1})\}$$
$$+ (1-\alpha) \cdot \frac{1}{D}\{(B\Sigma^{-1}\mathbb{1} - A\Sigma^{-1}\mu) + \mu_q(C\Sigma^{-1}\mu - A\Sigma^{-1}\mathbb{1})\}$$

$$= \alpha w_p + (1-\alpha) w_q$$

So, the portfolio $\alpha w_p + (1-\alpha) w_q$ is efficient for $\forall\, \alpha \in \mathbb{R}$.

(2) for any efficient portfolio w_0, there exist a value $\alpha \in \mathbb{R}$, such that $w_0 = \alpha w_p + ($

Let μ_0: the expected return of the portfolio w_0

we construct: $\alpha = \frac{\mu_0 - \mu_q}{\mu_p - \mu_q}$ $(\mu_p \neq \mu_q)$

$$\alpha\mu_p + (1-\alpha)\mu_q = \frac{\mu_0 - \mu_q}{\mu_p - \mu_q} \cdot \mu_p + \left(1 - \frac{\mu_0 - \mu_q}{\mu_p - \mu_q}\right)\mu_q$$

$$= \frac{\mu_0 - \mu_q}{\mu_p - \mu_q}\mu_p + \frac{\mu_p - \mu_0}{\mu_p - \mu_q} \cdot \mu_q$$

$$= \frac{\mu_0\mu_p - \cancel{\mu_p\mu_q} + \cancel{\mu_p\mu_q} - \mu_0\mu_q}{\mu_p - \mu_q} = \frac{\mu_0(\mu_p - \mu_q)}{\mu_p - \mu_q} = \mu$$

From part (a), we know there must have $w_0 = \alpha w_p + (1-\alpha) w_q$

Corresponding to $\mu_0 = \alpha\mu_p + (1-\alpha)\mu_q$

So,

$$\alpha = \frac{\mu_0 - \mu_q}{\mu_p - \mu_q} \quad (\mu_p \neq \mu_q)$$

or $$= \frac{(w_0^T - w_q^T)\mu}{(w_p^T - w_q^T)\mu} \quad (w_p \neq w_q)$$

a portfolio of n assets

$V = Cov(X)$, $X = (x_1, \dots, x_n)^T$

$\lambda_1 \geq \dots \geq \lambda_p$: eigenvalues of V

$a_1, \dots, a_p$: the corresponding eigenvectors with $\|a_i\| = 1$ for $i = 1, \dots, p$

We know. $I = a_1 a_1^T + \dots + a_p a_p^T$ (since $a_1, \dots, a_p$ are diagonal vectors &

fo

$V = VI = V(a_1 a_1^T + \dots + a_p a_p^T)$

$= V a_1 a_1^T + \dots + V a_p a_p^T$

Since $\underset{\sim}{V}\underset{\sim}{a} = \lambda \underset{\sim}{a}$,

So, $V = V a_1 a_1^T + \dots + V a_p a_p^T$

$= \lambda_1 a_1 a_1^T + \dots + \lambda_p a_p a_p^T$