Solutions to Assignments 1 of AMS 516

Problem 2.4 Solution:

(a) Define $B = \begin{pmatrix} \widetilde{A_{11}}^{-1} & -\widetilde{A_{11}}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}\widetilde{A_{11}}^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}\widetilde{A_{11}}^{-1}A_{12}A_{22}^{-1} \end{pmatrix}$

Suppose that $AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

We have $a_{11} = A_{11}\widetilde{\mathbf{A_{11}}}^{-\mathbf{1}} - A_{12}A_{22}^{-1}A_{21}\widetilde{\mathbf{A_{11}}}^{-\mathbf{1}} = I_m$

$a_{12} = -A_{11}\widetilde{A_{11}}^{-1}\mathbf{A_{12}A_{22}^{-1}} + \mathbf{A_{12}}(\mathbf{A_{22}^{-1}} + A_{22}^{-1}A_{21}\widetilde{A_{11}}^{-1}\mathbf{A_{12}A_{22}^{-1}}) = 0$

$a_{21} = \mathbf{A_{21}}\widetilde{\mathbf{A_{11}}}^{-\mathbf{1}} - A_{22}A_{22}^{-1}\mathbf{A_{21}}\widetilde{\mathbf{A_{11}}}^{-\mathbf{1}} = 0$

$a_{22} = -A_{21}\widetilde{A_{11}}^{-1}\mathbf{A_{12}A_{22}^{-1}} + A_{22}(A_{22}^{-1} + A_{22}^{-1}A_{21}\widetilde{A_{11}}^{-1}\mathbf{A_{12}A_{22}^{-1}}) = I_{n-m}$

Hence $AB = \begin{pmatrix} I_m & 0 \\ 0 & I_{n-m} \end{pmatrix} = I_n \rightarrow B = A^{-1}$

(b) Since A_{22} is non-singular, we have that A_{22}^{-1} exists.

$\det(A) = \det\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \det\begin{pmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ A_{21} & A_{22} \end{pmatrix}$

$= \det(A_{11} - A_{12}A_{22}^{-1}A_{21}) * \det(A_{22})$

(c) Define $Y^* = \begin{pmatrix} Y_1^* \\ Y_2^* \end{pmatrix} = \begin{pmatrix} Y_1 - \mu_1 \\ Y_2 - \mu_2 \end{pmatrix}$ we have $Y^* \sim N(0, V)$

Define $P = \begin{pmatrix} I & -V_{12}V_{22}^{-1} \\ 0 & I \end{pmatrix}$ we have $PY^* = \begin{pmatrix} Y_1^* - V_{12}V_{22}^{-1}Y_2^* \\ Y_2^* \end{pmatrix}$

Since $PVP^T = \begin{pmatrix} V_{11} - V_{12}V_{22}^{-1}V_{21} & 0 \\ 0 & V_{22} \end{pmatrix}$

We have $PY^* \sim N(0, PVP^T) = N(0, \begin{pmatrix} V_{11} - V_{12}V_{22}^{-1}V_{21} & 0 \\ 0 & V_{22} \end{pmatrix})$

Hence $Y_1^* - V_{12}V_{22}^{-1}Y_2^*$ and Y_2^* are independent.

$f(Y_1^* - V_{12}V_{22}^{-1}Y_2^* | Y_2^*) = f(Y_1^* - V_{12}V_{22}^{-1}Y_2^*) \sim N(0, V_{11} - V_{12}V_{22}^{-1}V_{21})$

$f(Y_1^* | Y_2^*) \sim N(V_{12}V_{22}^{-1}Y_2^*, V_{11} - V_{12}V_{22}^{-1}V_{21})$

$f_{Y_1|Y_2}(y_1|y_2) \sim N(\mu_1 + V_{12}V_{22}^{-1}(y_2 - \mu_2), V_{11} - V_{12}V_{22}^{-1}V_{21})$

Problem 2.5 Solution:

$$f(X_1, X_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right\} \leftrightarrow \binom{X_1}{X_2} \sim N(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix})$$

We have $X_1 - \rho X_2$ and X_2 are independent.

$$\begin{aligned} \mathrm{Cov}(X_1^2, X_2^2) &= E(X_1^2 X_2^2) - E(X_1^2)E(X_2^2) \\ &= E[(X_1 - \rho X_2)^2 X_2^2 + 2\rho(X_1 - \rho X_2)X_2^3 + \rho^2 X_2^4] - E(X_1^2)E(X_2^2) \\ &= E(X_1 - \rho X_2)^2 EX_2^2 + 2\rho E(X_1 - \rho X_2)EX_2^3 + \rho^2 EX_2^4 - E(X_1^2)E(X_2^2) \\ &= 2\rho^2 \end{aligned}$$

$\mathrm{Var}(X_1^2) = EX_1^4 - EX_1^2 EX_1^2 = 3 - 1 * 1 = 2$, $\mathrm{Var}(X_2^2) = 2$

Hence $\mathrm{Corr}(X_1^2, X_2^2) = \frac{\mathrm{Cov}(X_1^2, X_2^2)}{\sqrt{\mathrm{Var}(X_1^2)}\sqrt{\mathrm{Var}(X_2^2)}} = \rho^2$

Problem 2.6 Solution:

(a) Define $\bar{x} = (x_1, x_2, x_3, x_4, x_5)^T = \frac{\sum_{i=1}^n x_i}{n}$

We have $x_1 = \frac{\sum_{i=1}^n U_i}{n}, x_2 = \frac{\sum_{i=1}^n V_i}{n}, x_3 = \frac{\sum_{i=1}^n U_i^2}{n}, x_4 = \frac{\sum_{i=1}^n U_i^2}{n}, x_5 = \frac{\sum_{i=1}^n U_i V_i}{n}$

$$S_{UV} = \frac{\sum_{i=1}^n (U_i - \bar{U})(V_i - \bar{V})}{n} = \frac{\sum_{i=1}^n U_i V_i}{n} - \frac{\sum_{i=1}^n U_i}{n} * \frac{\sum_{i=1}^n V_i}{n} = x_5 - x_1 x_2$$

$$S_U^2 = \frac{\sum_{i=1}^n (U_i - \bar{U})^2}{n} = \frac{\sum_{i=1}^n U_i^2}{n} - \left(\frac{\sum_{i=1}^n U_i}{n}\right)^2 = x_3 - x_1^2, S_V^2 = x_4 - x_2^2$$

Hence $\hat{\rho} = \frac{S_{UV}}{\sqrt{S_U^2 S_V^2}} = \frac{x_5 - x_1 x_2}{(x_3 - x_1^2)^{\frac{1}{2}}(x_4 - x_2^2)^{\frac{1}{2}}} \rightarrow \hat{\rho} = g(\bar{x})$

(b) Define $\overline{x_0} = E(\bar{x}) = (0,0,1,1,\rho)^T$ $\Sigma = \mathrm{Cov}\{U, V, U^2, V^2, UV\}$

$$\text{We have } \Sigma = \begin{pmatrix} 1 & \rho & & & \\ \rho & 1 & & & \\ & & 2 & 2\rho^2 & 2\rho \\ & & 2\rho^2 & 2 & 2\rho \\ & & 2\rho & 2\rho & 1+\rho^2 \end{pmatrix}$$

According to the Central Limit Theorem, we have

$$\sqrt{n}(\bar{x} - \overline{x_0}) \rightarrow N(0, \Sigma) \text{ when } n \rightarrow \infty$$

By applying the delta method, we have

$$\sqrt{n}(g(\bar{x}) - g(\overline{x_0})) \rightarrow N(0, \nabla g(\overline{x_0})^T * \Sigma * \nabla g(\overline{x_0})) \text{ when } n \rightarrow \infty$$

Where $g(\bar{x}) = \hat{\rho}, g(\overline{x_0}) = \rho, \nabla g(\overline{x_0}) = \left(0,0,\frac{\rho}{2},\frac{\rho}{2},1\right)^T$

Hence $\sqrt{n}(\hat{\rho} - \rho) \rightarrow N(0, (1-\rho^2)^2)$ when $n \rightarrow \infty$

Problem 2.9 Solution:

(a) $E(\nabla \log f_\theta(X_1,\cdots,X_n) = E\left(\frac{\nabla f_\theta(X_1,\cdots,X_n)}{f_\theta(X_1,\cdots,X_n)}\right) = \int \frac{\nabla f_\theta(X_1,\cdots,X_n)}{f_\theta(X_1,\cdots,X_n)} * f_\theta(X_1,\cdots,X_n) dX_1 \cdots dX_n$

$$= \int \nabla f_\theta(X_1,\cdots,X_n)\, dX_1 \cdots dX_n = \nabla \int f_\theta(X_1,\cdots,X_n)\, dX_1 \cdots dX_n = \nabla 1 = 0$$

(b) $\mathrm{Cov}\big(\nabla \log f_\theta(X_1,\cdots,X_n)\big) = E\big(\nabla l(\theta)\big)^2 - \big[E\big(\nabla \log f_\theta(X_1,\cdots,X_n)\big)\big]^2 = E\big(\nabla l(\theta)\big)^2$

$E\big(\nabla l(\theta)\big)^2 - E\big(-\nabla^2 l(\theta)\big) = E\big(\nabla l(\theta) * \nabla l(\theta) + \nabla^2 l(\theta)\big) = E\big(\nabla * \nabla l(\theta)\big)$

$$= \nabla * E\big(\nabla l(\theta)\big) = \nabla 0 = 0$$

Problem 4.1 Solution:

(a) Define $Y = (y_1,\cdots y_n)^T, X = (x_1,\cdots,x_n)^T, \mathbf{1} = (1,\cdots,1)^T$

We have $y_i = \alpha + \beta x_i + \varepsilon_i \rightarrow Y - \alpha\mathbf{1} - \beta X \sim N(0,\sigma^2 I)$

$$l(\alpha,\beta,\sigma^2) = \log L(\alpha,\beta,\sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) + \sum_{i=1}^{n}\{-\frac{1}{2\sigma^2}(y_i - \alpha - \beta x_i)^2\}$$

From $\begin{cases} \frac{\partial l}{\partial \alpha} = 0 \\ \frac{\partial l}{\partial \beta} = 0 \\ \frac{\partial l}{\partial \sigma^2} = 0 \end{cases} \rightarrow \begin{cases} \sum_{i=1}^{n} \frac{y_i - \alpha - \beta x_i}{\sigma^2} = 0 \\ \sum_{i=1}^{n} \frac{x_i(y_i - \alpha - \beta x_i)}{\sigma^2} = 0 \\ -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(y_i - \alpha - \beta x_i)^2 = 0 \end{cases}$

Hence $\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}, \hat{\beta} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}, \widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}[(y_i - \bar{y}) - \hat{\beta}(x_i - \bar{x})]$

Where $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}, \bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}$

(b) Since V is the correlation matrix, we have the decomposition of V:

$V = AA^T$ Where A is non-singular

In this problem, we have $Y - \alpha\mathbf{1} - \beta X \sim N(0,\sigma^2 V)$

Hence $A^{-1}(Y - \alpha\mathbf{1} - \beta X) \sim N(0, A^{-1}\sigma^2 V (A^{-1})^T) = N(0,\sigma^2 I)$

$$l(\alpha,\beta,\sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}[A^{-1}(Y - \alpha\mathbf{1} - \beta X)]^T[A^{-1}(Y - \alpha\mathbf{1} - \beta X)]$$

$$= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(Y - \alpha\mathbf{1} - \beta X)^T V^{-1}(Y - \alpha\mathbf{1} - \beta X)$$

From $\begin{cases} \frac{\partial l}{\partial \alpha} = 0 \\ \frac{\partial l}{\partial \beta} = 0 \\ \frac{\partial l}{\partial \sigma^2} = 0 \end{cases} \rightarrow \begin{cases} \hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} \\ \hat{\beta} = \frac{(X - \bar{X})^T V^{-1}(Y - \bar{Y})}{(X - \bar{X})^T V^{-1}(X - \bar{X})} \\ \widehat{\sigma^2} = \big[(Y - \bar{Y}) - \hat{\beta}(X - \bar{X})\big]^T V^{-1}[(Y - \bar{Y}) - \hat{\beta}(X - \bar{X})] \end{cases}$

Where $\bar{X} = \bar{x} * \mathbf{1}, \bar{Y} = \bar{y} * \mathbf{1}$

Problem 4.2 Solution:

$$l(\theta) = \log f(y;\theta,\varphi) = \frac{y\theta + b(\theta)}{\varphi} + c(y,\varphi)$$

From 2.9(a) $E\left(\frac{\partial l}{\partial \theta}\right) = 0$ we have $\frac{E(y)-b'(\theta)}{\varphi} = 0$ hence $E(y) = b'(\theta)$

From 2.9(b) $E\left(-\frac{\partial^2 l}{\partial \theta^2}\right) = E\left(\frac{\partial l}{\partial \theta}\right)^2$ we have $\frac{b''(\theta)}{\varphi} = E\left(\frac{y-b'(\theta)}{\varphi}\right)^2$

Hence $E(y^2) - 2E(y)b'(\theta) + b'(\theta)^2 = \varphi b''(\theta)$

$$Var(y) = E(y^2) - [E(y)]^2 = \varphi b''(\theta)$$