Yunting Zhang
1077428865

①

Hwk#2
3.1, 3.2, 3.3, 3.6, Prove (3.15), Prove P54 (a) ~ (c)

3.1 Prove (3.19)

$$W_{eff} = \frac{(\mu^* - r_f)}{(\mu - r_f \mathbb{1})^T \Sigma^{-1} (\mu - r_f \mathbb{1})} \Sigma^{-1} (\mu - r_f \mathbb{1})$$

$\min_{W} W^T \Sigma W$ subject to $W^T \mu + (1 - W^T \mathbb{1}) r_f = \mu^*$

which $r = (R_1, \dots, R_p)^T$ denote the vector of returns of p assets

$\mathbb{1} = (1, \dots, 1)^T$

$W = (W_1, \dots, W_p)^T$

$\mu = (\mu_1, \dots, \mu_p)^T = (E(R_1), \dots, E(R_p))^T$

$\Sigma = (Cov(R_i, R_j))$, $1 \le i \le j \le p$

We know the Lagrangian is:

$$L = W^T \Sigma W + 2\lambda \{ W^T \mu + (1 - W^T \mathbb{1}) r_f - \mu^* \}$$

$$\begin{cases} \frac{\partial L}{\partial W} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases} \Rightarrow \begin{cases} \Sigma W + \lambda (\mu - r_f \mathbb{1}) = 0 & \text{— ①} \\ W^T \mu + (1 - W^T \mathbb{1}) r_f = \mu^* & \text{— ②} \end{cases}$$

From ①,

$$\Sigma^{-1} \Sigma W + \lambda \Sigma^{-1} (\mu - r_f \mathbb{1}) = 0$$

$$W = -\lambda \Sigma^{-1} (\mu - r_f \mathbb{1}) \quad \text{— ③}$$

Substitute ③ into ②,

$$-\lambda (\mu - r_f \mathbb{1})^T (\Sigma^{-1})^T \mu + [1 + \lambda (\mu - r_f \mathbb{1})^T (\Sigma^{-1})^T \mathbb{1}] r_f = \mu^*$$

$$-\lambda (\mu - r_f \mathbb{1})^T (\Sigma^{-1})^T \mu + r_f + \lambda (\mu - r_f \mathbb{1})^T (\Sigma^{-1})^T r_f = \mu^*$$

$$\lambda ((\mu - r_f \mathbb{1})^T (\Sigma^{-1})^T (r_f \mathbb{1} - \mu) = \mu^* - r_f$$

So,

$$\lambda = \frac{\mu^* - r_f}{(\mu - r_f \mathbb{1})^T (\Sigma^{-1})^T (r_f \mathbb{1} - \mu)} \quad \text{— ④}$$

Since Σ is symmetric, $(\Sigma^{-1})^T = (\Sigma^T)^{-1} = \Sigma^{-1}$

Substitute ④ into ③,

$$W_{eff} = -\frac{\mu^* - r_f}{(\mu - r_f \mathbb{1})^T (\Sigma^{-1})^T (r_f \mathbb{1} - \mu)} \cdot (\Sigma^{-1}) \cdot (\mu - r_f \mathbb{1})$$

$$= \frac{\mu^* - r_f}{(\mu - r_f \mathbb{1})^T \Sigma^{-1} (\mu - r_f \mathbb{1})} \cdot (\Sigma^{-1}) (\mu - r_f \mathbb{1})$$

3.2 Prove 3.27 and 3.28

Let y_t be a $q \times 1$ vector of excess returns on q assets

Let x_t be the excess return on the market portfolio at time t.

The capital asset pricing model is associated with the null hypothesis

$H_0: \alpha = 0$ in the regression model: $y_t = \alpha + x_t\beta + \epsilon_t$, $1 \le t \le n$.

where $E(\epsilon_t) = 0$, $Cov(\epsilon_t) = 0$ } ϵ_t are iid normal

$E(x_t\epsilon_t) = 0$ $\Rightarrow$ ϵ_t are indep. of the market excess returns x_t.

Let $\bar{x} = n^{-1}\sum_{t=1}^{n} x_t$, $\bar{y} = n^{-1}\sum_{t=1}^{n} y_t$, $\hat{V} = n^{-1}\sum_{t=1}^{n}(y_t - \hat{\alpha} - \hat{\beta}x_t)(y_t - \hat{\alpha} - \hat{\beta}x_t)^T$

$$\hat{\beta} = \frac{\sum_{t=1}^{n}(x_t - \bar{x})y_t}{\sum_{t=1}^{n}(x_t - \bar{x})^2}, \quad \hat{\alpha} = \bar{y} - \bar{x}\hat{\beta}$$

Prove $\hat{\alpha} \sim N(\alpha, (\frac{1}{n} + \frac{\bar{x}^2}{\sum_{t=1}^{n}(x_t - \bar{x})^2})V)$ — ②

$\hat{\beta} \sim N(\beta, \frac{V}{\sum_{t=1}^{n}(x_t - \bar{x})^2})$ — ①

$n\hat{V} \sim W_q(V, n-2)$ — ③

For ① Since $\mathcal{E}_t$ are iid normal

$\mathcal{E}_t \sim N(\underset{\sim}{0}, V)$ and given $(X_1, \dots, X_n)$

$Cov(y_{ti}, y_{tj}) = Cov(\alpha_i + x_t\beta_i + \mathcal{E}_{ti}, \alpha_j + x_t\beta_j + \epsilon_{tj})$

$= Cov(\mathcal{E}_{ti}, \mathcal{E}_{tj}) = V_{ij}$

let $A = \sum_{t=1}^{n}(X_t - \bar{X})^2$

$$Cov(\hat{\beta}_i, \hat{\beta}_j) = Cov\left(\frac{\sum_{t=1}^{n}(X_t - \bar{X})y_{ti}}{A}, \frac{\sum_{t=1}^{n}(X_t - \bar{X})y_{tj}}{A}\right)$$

$$= \frac{1}{A^2} Cov\left(\sum_{t=1}^{n}(X_t - \bar{X})y_{ti}, \sum_{t=1}^{n}(X_t - \bar{X})y_{tj}\right)$$

$$= \frac{1}{A^2}\left(\sum_{p=1}^{n}\sum_{q=1}^{n}(X_p - \bar{X})(X_q - \bar{X})Cov(y_{pi}, y_{qj})\right)$$

$$= \frac{1}{A^2}\sum_{t=1}^{n}(X_t - \bar{X})^2 V_{ij} = \frac{V_{ij}}{\sum_{t=1}^{n}(X_t - \bar{X})^2}$$

$$E[\hat{\beta}_i] = E\left[\frac{\sum_{t=1}^{n}(X_t - \bar{X})(\alpha_i + X_t\beta_i + \mathcal{E}_{ti})}{\sum_{t=1}^{n}(X_t - \bar{X})^2}\right]$$

$$= \frac{\sum_{t=1}^{n}(X_t - \bar{X})X_t}{\sum_{t=1}^{n}(X_t - \bar{X})^2}\beta_i = \beta_i$$

Since $\mathcal{E}_t$ are iid normal and given $(X_1, \dots X_n)$,

then, y_t is also normal, so, $\hat{\beta}$ is normal $\sim N(\beta, \frac{V}{\sum^{n}(X_t - \bar{X})^2}$

For ②

$$E[\hat{\alpha}_i] = E[\bar{y} - \bar{x}\hat{\beta}_i] = E[\frac{1}{n}\sum_{t=1}^{n} y_{ti} - \frac{1}{n}\sum_{t=1}^{n} x_{ti}\hat{\beta}_i]$$

Since $E[y_{ti}] = E[\alpha_i + x_t\beta_i + \varepsilon_t] = \alpha_i + \beta_i\bar{x}$

So, $E[\hat{\alpha}_i] = E[\alpha_i + \beta_i\bar{x} - \beta_i\bar{x}] = \alpha_i$

$Cov(\hat{\underset{\sim}{\alpha}}) = Cov(\underset{\sim}{\bar{y}} - \bar{x}\hat{\beta})$

$Cov(\hat{\alpha}_i, \hat{\alpha}_j) = Cov(\bar{y}_i - \bar{x}\hat{\beta}_i, \bar{y}_j - \bar{x}\hat{\beta}_j) = Cov(\bar{y}_i, \bar{y}_j) - Cov(y_i, \bar{x}\hat{\beta}_j) - Cov(\bar{x}\hat{\beta}_i,$
$+ Cov(\bar{x}\hat{\beta}_i, \bar{x}\hat{\beta}_j)$

$Cov(\bar{y}_i, \bar{y}_j) = Cov(\frac{1}{n}\sum_{t=1}^{n} y_{ti}, \frac{1}{n}\sum_{t=1}^{n} y_{tj}) = \frac{1}{n^2}\sum_{t=1}^{n} Cov(y_{ti}, y_{tj}) = \frac{1}{n^2}\sum_{t=1}^{n} V_{ij} = \frac{1}{n}V_{ij}$

$Cov(\bar{y}_i, \bar{x}\hat{\beta}_j) = Cov(\frac{1}{n}\sum_{t=1}^{n} y_{ti}, \bar{x}\frac{\sum_{t=1}^{n}(x_t - \bar{x})y_{tj}}{\sum_{t=1}^{n}(x_t-\bar{x})^2}) = \frac{\bar{x}}{nA} Cov(\sum_{t=1}^{n} y_{ti}, \sum_{t=1}^{n}(x_t - \bar{x})y_t$

$= \frac{\bar{x}}{nA}\sum_{t=1}^{n}(x_t - \bar{x})Cov(y_{ti}, y_{tj}) = \frac{\bar{x}}{nA}\sum_{t=1}^{n}(x_t - \bar{x})V_{ij} = 0,$ where $A = \sum_{t=1}^{n}(x_t - \bar{x})^2$

Similarly, $Cov(\bar{x}\hat{\beta}_i, \bar{y}_j) = 0$

$Cov(\bar{x}\hat{\beta}_i, \bar{x}\hat{\beta}_j) = \bar{x}^2 Cov(\hat{\beta}_i, \hat{\beta}_j) = \bar{x}^2 \frac{V_{ij}}{\sum_{t=1}^{n}(x_t-\bar{x})^2}$

$Cov(\hat{\alpha}_i, \hat{\alpha}_j) = \frac{1}{n}V_{ij} + \frac{\bar{x}^2}{\sum_{t=1}^{n}(x_t-\bar{x})^2}V_{ij}$

So, $\hat{\underset{\sim}{\alpha}} \sim N(\underset{\sim}{\alpha}, \frac{1}{n} + (\frac{\bar{x}^2}{\sum_{t=1}^{n}(x_t-\bar{x})^2})V)$

For ③ To estimate the MLE of α and β.

there're two constraint equations

and $\{(y_t - \hat{\alpha} - x_t\hat{\beta})\} \overset{iid}{\sim} N(0, V)$

So, $n\hat{V} = \sum_{t=1}^{n}(y_t - \hat{\alpha} - \hat{\beta}x_t)(y_t - \hat{\alpha} - \hat{\beta}x_t)^T \sim W_q(V, n-2)$

Prove 3.28 $\left(\frac{n-q-1}{q}\right)\hat{\alpha}^T\hat{V}\hat{\alpha} \Big/ \left\{1 + \frac{\bar{x}^2}{n^{-1}\sum_{t=1}^{n}(x_t-\bar{x})^2}\right\} \sim F_{q, n-q-1}$ under H_0.

Let $M = (\frac{1}{n} + \frac{\bar{x}^2}{\sum_{t=1}^{n}(x_t-\bar{x})^2})$, under H_0,

$\hat{\alpha} \sim N(0, M\underset{\sim}{V})$, so, $\frac{1}{\sqrt{M}}\hat{\alpha} \sim N(0, \underset{\sim}{V})$

$W = n\hat{V} \sim W_q(V, n-2)$

We know $t = \left(\frac{W}{n-2}\right)^{-\frac{1}{2}}\frac{1}{\sqrt{M}}\hat{\alpha}$ has q variates t-distribution with $(n-2)$ df

$$\frac{n-2-q+1}{(n-2)q}t^T t = \frac{n-q-1}{(n-2)q}\frac{1}{\sqrt{M}}\hat{\alpha}^T\left[\left(\frac{W}{n-2}\right)^{-\frac{1}{2}}\right]^T\left(\frac{W}{n-2}\right)^{-\frac{1}{2}}\frac{1}{\sqrt{M}}\hat{\alpha}$$

$$= \frac{n-q-1}{(n-2)q}\frac{1}{M}\hat{\alpha}^T\left(\frac{n\hat{V}}{n-2}\right)^{-1}\hat{\alpha}$$

$$= \frac{n-q-1}{nq}\frac{1}{M}\hat{\alpha}^T\hat{V}^{-1}\hat{\alpha} \sim F_{q, n-q-1} \text{ under } H_0$$

3.3. Let r_{0t} be the return of a stock index at time t. Sharpe's single-index model assumes that the log returns of the n stocks in the index are generated by $r_{it} = \alpha_i + \beta_i r_{0t} + \varepsilon_{it}$, $1 \le i \le p$, where ε_{it} is uncorrelated with r_{0t} and $\mathrm{Cov}(\varepsilon_{it}, \varepsilon_{jt}) = \sigma^2 \mathbb{1}\{i=j\}$. The model also assumes that $(r_{0t}, \dots, r_{pt})$, $1 \le t \le n$, are iid vectors

(a) Suppose $\mathrm{Var}(r_{0t}) = \sigma_0^2$. Show that the covariance matrix $\mathbb{F} = (f_{ij})$ of the log return of the n stocks under the single-index model is given by $F = \sigma_0^2 \beta\beta^T + \sigma^2 I$, where $\beta = (\beta_1, \dots, \beta_p)^T$

$$
\begin{aligned}
f_{ij} = \mathrm{Cov}(r_{it}, r_{jt}) &= \mathrm{Cov}(\alpha_i + \beta_i r_{0t} + \varepsilon_{it},\ \alpha_j + \beta_j r_{0t} + \varepsilon_{jt}) \\
&= \mathrm{Cov}(\beta_i r_{0t} + \varepsilon_{it},\ \beta_j r_{0t} + \varepsilon_{jt}) \\
&= \mathrm{Cov}(\beta_i r_{0t}, \beta_j r_{0t}) + \mathrm{Cov}(\varepsilon_{it}, \varepsilon_{jt}) \\
&= \beta_i \beta_j \mathrm{Var}(r_{0t}) + \sigma^2 \mathbb{1}_{\{i=j\}}
\end{aligned}
$$

So, $\mathbb{F} = (f_{ij}) = \sigma_0^2 \beta\beta^T + \sigma^2 \mathbb{1}$

(b) Let $\sigma_{ij} = \mathrm{Cov}(r_{it}, r_{jt})$ and $\Sigma = (\sigma_{ij})_{1 \le i,j \le p}$. Let $\mathbb{S} = (S_{ij})$ be the sample covariance matrix based on $(r_{it}, \dots, r_{pt})^T$, $1 \le t \le n$. Let $R(\alpha) = \alpha\mathbb{F} + (1-\alpha)\mathbb{S}$. Consider the quadratic loss function $L(\alpha) = \|\cdot\|^2$, where $\|A\|$ is the Frobenius norm of a square matrix A defined by $\|A\|^2 = \mathrm{tr}(A^T A)$. Show that the minimizer α^* of $E[L(\alpha)]$ is given by

$$
\alpha^* = \frac{\sum_{i=1}^{p}\sum_{j=1}^{p}[\mathrm{Var}(S_{ij}) - \mathrm{Cov}(f_{ij}, S_{ij})]}{\sum_{i=1}^{p}\sum_{j=1}^{p}[\mathrm{Var}(f_{ij} - S_{ij}) + (E(f_{ij}) - \sigma_{ij})^2]}
$$

$$
L(\alpha) = \|\alpha\mathbb{F} + (1-\alpha)\mathbb{S} - \Sigma\|^2 = \mathrm{tr}((\alpha\mathbb{F} + (1-\alpha)\mathbb{S} - \Sigma)(\alpha F + (1-\alpha)\mathbb{S} - \Sigma))
$$

since $\mathbb{F}, \mathbb{S}, \Sigma$ are symmetric,

$$
L(\alpha) = \sum_{i=1}^{p}\sum_{j=1}^{p}(\alpha f_{ij} + (1-\alpha)S_{ij} - \sigma_{ij})^2
$$

$$
E[L(\alpha)] = \sum_{i=1}^{p}\sum_{j=1}^{p} E[(\alpha f_{ij} + (1-\alpha)S_{ij} - \sigma_{ij})^2]
$$

Since σ_{ij} is a constant,

$$
\begin{aligned}
&= \sum_{i=1}^{p}\sum_{j=1}^{p}\left[\mathrm{var}(\alpha f_{ij} + (1-\alpha)S_{ij} - \sigma_{ij}) + \left(E[\alpha f_{ij} + (1-\alpha)S_{ij} - \sigma_{ij}]\right)^2\right] \\
&= \sum_{i=1}^{p}\sum_{j=1}^{p}\Big[\alpha^2 \mathrm{Var}(f_{ij}) + (1-\alpha)^2 \mathrm{Var}(S_{ij}) + 2\alpha(1-\alpha)\mathrm{Cov}(f_{ij}, S_{ij}) \\
&\qquad + \alpha^2 (E[f_{ij}])^2 + (1-\alpha)^2 (E[S_{ij}])^2 + 2\alpha(1-\alpha)E[f_{ij}]E[S_{ij}] \\
&\qquad - 2\alpha\sigma_{ij}E[f_{ij}] - 2(1-\alpha)\sigma_{ij}E[S_{ij}]\Big]
\end{aligned}
$$

Since $E[S_{ij}] = \sigma_{ij}$,

$$
\begin{aligned}
&= \sum_{i=1}^{p}\sum_{j=1}^{p}\Big[\alpha^2 \mathrm{Var}(f_{ij}) + (1-\alpha)^2 \mathrm{Var}(S_{ij}) + 2\alpha(1-\alpha)\mathrm{Cov}(f_{ij}, S_{ij}) \\
&\qquad + \alpha^2 (E[f_{ij}])^2 + (1-\alpha)^2 \sigma_{ij}^2 + 2\alpha(1-\alpha)E[f_{ij}]\sigma_{ij} \\
&\qquad - 2\alpha\sigma_{ij}E[f_{ij}] - 2(1-\alpha)\sigma_{ij}^2\Big]
\end{aligned}
$$

$2(1-\alpha)+2\alpha(-1) = 2-2\alpha-2\alpha = 2(1-\alpha$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p}\left[\alpha^2 Var(f_{ij}) + (1-\alpha)^2 Var(S_{ij}) + 2\alpha(1-\alpha) Cov(f_{ij}, S_{ij}) + \alpha^2 (E(f_{ij}) - \sigma_{ij})^2\right]$$

differentiate with respect to α:

$$\frac{\partial E[L(\alpha)]}{\partial \alpha} = 2\left[\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\alpha Var(f_{ij}) - (1-\alpha)Var(S_{ij}) + (1-2\alpha)Cov(f_{ij}, S_{ij}) + \alpha(E(f_{ij}) - \sigma_{ij})^2\right)\right]$$

let $\frac{\partial E[L(\alpha)]}{\partial \alpha} = 0$, then,

$$\alpha^* = \frac{\sum_{i=1}^{p}\sum_{j=1}^{p}[Var(S_{ij}) - Cov(f_{ij}, S_{ij})]}{\sum_{i=1}^{p}\sum_{j=1}^{p}[Var(f_{ij} - S_{ij}) + (E[f_{ij}] - \sigma_{ij})^2]}$$

By taking the 2nd derivative of $E[L(\alpha)]$ respect to α,

$$\frac{\partial^2 E[L(\alpha)]}{\partial^2 \alpha} = 2\sum_{i=1}^{p}\sum_{j=1}^{p}[Var(f_{ij} - S_{ij}) + (E[f_{ij}] - \sigma_{ij})^2] \geq 0$$

So, α^* is the minimizer of $E[L(\alpha)]$

Prove 3.15 The covariance of the returns r_p and r_q is given by

$$Cov(r_p, r_q) = \frac{C}{D}(\mu_p - \frac{A}{C})(\mu_q - \frac{A}{C}) + \frac{1}{C}$$

where $A = \mu^T \Sigma^{-1} \mathbb{1} = \mathbb{1}^T \Sigma^{-1} \mu$

$B = \mu^T \Sigma^{-1} \mu$

$C = \mathbb{1}^T \Sigma^{-1} \mathbb{1}$

$D = BC - A^2$

We know $r_p = w_p^T \cdot r$, $r_q = w_q^T \cdot r$

$Cov(r_p, r_q) = Cov(w_p^T r, w_q^T \cdot r) = Cov(r^T w_p, w_q^T \cdot r)$

$w_p = \{B\Sigma^{-1}\mathbb{1} - A\Sigma^{-1}\mu + \mu_p(C\Sigma^{-1}\mu - A\Sigma^{-1}\mathbb{1})\}/D$

$= \frac{B - \mu_p A}{D}\Sigma^{-1}\mathbb{1} + \frac{\mu_p C - A}{D}\Sigma^{-1}\mu$

$w_q = \frac{B - \mu_q A}{D}\Sigma^{-1}\mathbb{1} + \frac{\mu_q C - A}{D}\Sigma^{-1}\mu$

$Cov(r^T\Sigma^{-1}\mathbb{1}, \mathbb{1}^T\Sigma^{-1}r) = Var(r^T\Sigma^{-1}\mathbb{1}) = \mathbb{1}^T \underbrace{\Sigma^{-1}\Sigma}_{I} \Sigma^{-1}\mathbb{1} = C$

$Cov(r^T\Sigma^{-1}\mu, \mu^T\Sigma^{-1}r) = Var(r^T\Sigma^{-1}\mu) = \mu^T \underbrace{\Sigma^{-1}\Sigma}_{I} \Sigma^{-1}\mu = B$

let $w_1 = \frac{\Sigma^{-1}\mathbb{1}}{C}$, $w_2 = \frac{\Sigma^{-1}\mu}{A}$

then,

$Cov(r^T\Sigma^{-1}\mathbb{1}, \mathbb{1}^T\Sigma^{-1}r) = AC\, Cov(r^T w_1, w_2^T r) = AC\, w_1^T \Sigma w_2 = \frac{AC\,\mathbb{1}^T\Sigma^{-1}\Sigma\Sigma^{-1}\mu}{AC}$

$= A$

So, $\mathrm{Cov}(r_p, r_q) = \mathrm{Cov}(r^T w_p,\ w_q r)$

$$= \frac{(B-\mu_p A)(B-\mu_q A)}{D^2} C + \frac{(\mu_p C - A)(\mu_q - A)}{D^2} B$$

$$+ A \cdot \frac{(B-\mu_p A)(\mu_q C - A) + (\mu_p C - A)(B - \mu_q A)}{D^2}$$

$$= \frac{1}{D^2}\Big[C(B^2 - \mu_p AB - \mu_q AB + \mu_p\mu_q A^2) + B(\mu_p\mu_q C^2 - AC\mu_p - AC\mu_q + A^2) + A(BC\mu_q - AB - \mu_p\mu_q AC + \mu_p A^2 + BC\mu_p - AC\mu_p\mu_q - AB + A^2\mu_q)\Big]$$

$$= \frac{1}{D^2}\left[\mu_p\mu_q(BC^2 - A^2C) + B^2C - BA^2 + (\mu_p+\mu_q)A(A^2-BC)\right]$$

$$= \frac{1}{D^2}\left[\mu_p\mu_q(BC^2 - A^2C) + B^2C - BA^2 + (\mu_p+\mu_q)A(A^2-BC)\right]$$

$$= \frac{1}{D^2}\left[\mu_p\mu_q CD + BD + (\mu_p+\mu_q)AD\right]$$

$$= \frac{1}{D}\left[\mu_p\mu_q C + B + (\mu_p+\mu_q)A\right]$$

we know, $\frac{C}{D}(\mu_p - \frac{A}{C})(\mu_q - \frac{A}{C}) + \frac{1}{C} = \frac{C}{D}(\mu_p\mu_q - \frac{A}{C}\mu_p - \frac{A}{C}\mu_q + \frac{A^2}{C^2}) + \frac{D}{DC}$

$$= \frac{1}{D}\mu_p\mu_q C + \frac{A}{D}(\mu_p+\mu_q) + \frac{D+A^2}{DC}$$

$$= \frac{1}{D}\mu_p\mu_q C + \frac{A}{D}(\mu_p+\mu_q) + \frac{B}{D}$$

So, $\mathrm{Cov}(r_p, r_q) = \frac{C}{D}(\mu_p - \frac{A}{C})(\mu_q - \frac{A}{C}) + \frac{1}{C}$

Prove P84 (a), (b), (c)

(a) If $W \sim W_m(\Sigma, n)$, then $E[W] = n\Sigma$

We know $W \sim W_m(\Sigma, n)$,

there exists $y_1, \dots, y_n \overset{iid}{\sim} N(0, \Sigma)$ such that $W = \sum_{i=1}^n y_i y_i^T$

$E[W] = E[\sum_{i=1}^n y_i y_i^T] = \sum_{i=1}^n E[y_i y_i^T] = \sum_{i=1}^n Var(y_i) = n \cdot \Sigma$

(b) Let $W_1, \dots, W_k$ be independently distributed with $W_j \sim W_m(\Sigma, n_j)$, $j=1,\dots,k$. Then $\sum_{j=1}^k W_j \sim W_m(\Sigma, \sum_{j=1}^k n_j)$.

there exists $y_{j,1}, \dots, y_{j,n_j} \overset{iid}{\sim} N(0, \Sigma)$, $j=1,\dots,k$

$W = \sum_{j=1}^k W_j = \sum_{j=1}^k (\sum_{l=1}^{n_j} y_{jl} y_{jl}^T)$

$$= \sum_{l=1}^{\sum_{j=1}^k n_j} y_l y_l^T \sim W_m(\Sigma, \sum_{j=1}^k n_j)$$

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(c) Let $W \sim W_m(\Sigma, n)$ and A be a nonrandom $m\times m$ nonsingular matrix. Then $AWA^T \sim W_m(A\Sigma A^T, n)$. In particular, $a^T W a \sim (a^T\Sigma a)\chi^2_n$ for all nonrandom vectors $a \neq 0$.

$W \sim W_m(\Sigma, n)$, $W = \sum_{i=1}^n y_i y_i^T$, where $y_i \overset{iid}{\sim} N(0, \Sigma)$.

then $Ay_i \overset{iid}{\sim} N(0, A\Sigma A^T)$

$$AWA^T = A\left(\sum_{i=1}^n y_i y_i^T\right)A^T = \sum_{i=1}^n (Ay_i y_i^T A^T) = \sum_{i=1}^n (Ay_i)\cdot(Ay_i)^T \sim W(A\Sigma A^T,$$

when $A = a^T$, $a^T y_i \overset{iid}{\sim} N(0, a^T\Sigma a)$

$$a^T W a = a^T \sum_{i=1}^n y_i y_i^T a = \sum_{i=1}^n a^T y_i (a^T y_i)^T \sim (a^T\Sigma a)\chi^2_n$$

Problem3.6

a)

Point estimates of α and β:

α	0.0038	0.0047	0.0008	-0.0006	0.0089	-0.0054	0.0019	0.0026	0.0043	0.0039
β	1.3846	1.5314	0.8477	2.3238	1.6750	2.2328	0.8752	1.3479	1.4585	1.5676

95% CI for α and β:

α		β	
lower	upper	lower	upper
-0.0061	0.0138	0.8257	1.9435
-0.0050	0.0143	0.9907	2.0720
-0.0029	0.0045	0.6396	1.0558
-0.0123	0.0111	1.6697	2.9779
0.0008	0.0170	1.2209	2.1291
-0.0165	0.0057	1.6108	2.8548
-0.0053	0.0091	0.4720	1.2784
-0.0021	0.0074	1.0827	1.6131
-0.0015	0.0100	1.1347	1.7824
-0.0049	0.0127	1.0766	2.0586

Point estimates for Sharpe index and the Treynor index:

Sharpe	0.0544	0.0669	0.0249	- 0.0108	0.1461	- 0.0708	0.0373	0.0637	0.0906	0.0598
Treynor	0.0027	0.0029	0.0008	- 0.0004	0.0052	- 0.0026	0.0020	0.0018	0.0028	0.0024

b) use the bootstrap procedure in Section3.5 to estimate the standard errors of the point estimates of α, β and the Sharpe and Treynor indices:

run bootstrap 500 times,

and get the standard errors for the point estimates of α, β and the Sharpe and Treynor indices:

α	0.0051	0.0048	0.0019	0.0062	0.0041	0.0057	0.0039	0.0025	0.0028	0.0042
β	0.3317	0.2573	0.1298	0.3458	0.2685	0.4358	0.1903	0.1488	0.1711	0.2630
Sharpe	0.0801	0.0739	0.0716	0.0781	0.0720	0.0735	0.0811	0.0647	0.0646	0.0702
Treynor	0.0042	0.0034	0.0025	0.0029	0.0028	0.0028	0.005	0.0018	0.002	0.003

c) Test for each stock the null hypothesis α =0.

According to the 95% CI of α, we cannot reject the null hypothesis of each stock, expect the fifth stock-'DELL'.

d) Use the regression model(3.24) to test for the ten stocks the null hypothesis α =0.

We know that n=156, q=10.

The Test statistics is : $\dfrac{\frac{n-q-1}{q}\,\widehat{a^T}\,\hat{V}^{-1}\hat{a}}{1+\frac{\bar{x}^2}{n^{-1}\sum_{t=1}^{n}(x_t-\bar{x})^2}}$ $=1.0259 < F_{10,156-10-1,0.95}=1.60.$

So we do not reject the null hypothesis with 95% confidence level.

e) Perform a factor analysis on the excess returns of the ten stocks. Show the factor loading and rotated factor loadings. Explain your choice of the number of factors.

By PCA, we find the following factoring loadings:

-0.3726	0.1104	0.1006	-0.2885	0.7643	-0.3598	0.0767	-0.1649	0.008	-0.0947
-0.2885	-0.1982	0.5889	0.6543	0.0541	0.0461	0.2958	-0.0325	0.0041	0.0948
-0.0734	-0.0289	0.0609	0.0399	-0.1052	0.1579	-0.0858	-0.2750	0.4774	-0.801
-0.5256	0.679	-0.3351	0.2632	-0.2178	-0.0315	0.1046	0.1153	-0.0706	-0.0592
-0.3359	-0.2157	-0.0009	-0.3548	0.0446	0.4536	0.3022	0.6077	0.2152	-0.0084
-0.4898	-0.6205	-0.4045	0.1293	-0.1328	-0.2448	-0.3339	-0.0496	-0.0203	0.0528
-0.1132	0.1260	0.1317	0.1235	0.3001	0.5526	-0.6959	0.0502	-0.2376	0.0073
-0.1801	0.1037	0.0034	-0.1632	-0.0868	0.2619	-0.0202	-0.5151	0.5246	0.5596
-0.1982	-0.1208	0.0217	-0.288	-0.188	0.3135	0.3079	-0.4754	-0.6225	-0.1444
-0.243	0.118	0.5879	0.3878	-0.4494	-0.3235	-0.3255	0.1311	-0.0357	0.0073

Here's the cumulative explanatory power:

0.4398
0.5507
0.6532
0.7468
0.8230
0.8844
0.9372
0.9667
0.9862
1.0000

If we set the threshold= 0.8, then we choose 5 factors.

Here's the rotated factor loadings:

0.5022	0.2536	0.1691	0.3287	-0.1864	0.2694
0.3741	0.0908	-0.0774	0.3033	0.2971	0.2049
0.0925	0.1076	0.1767	0.1143	0.5973	0.1578
0.5210	0.1414	0.3096	0.2217	0.0682	0.1882
0.3250	0.8203	0.1278	0.1635	0.0585	0.1746
0.6909	0.3410	0.0719	0.0329	0.2288	-0.0050
0.1059	0.0259	0.0974	0.0446	0.1441	0.6110
1.1673	0.2794	0.8832	0.1723	0.2168	0.1797
0.1849	0.5759	0.3010	0.2207	0.1890	-0.1004
0.1428	0.1778	0.1537	0.5901	0.1362	0.0355

f) Taking February 2001 as the month t_0, split the sample into two subsamples [1,85] and [86,156]. Fit CAPM for each subsample:

the estimated point for β:

β from [1,85]	β from [86,156]
1.4569	1.4001
1.1768	1.9882
0.6727	1.0576
1.6295	3.2252
2.0793	1.0944

1.7552	2.7729
0.8403	0.9304
1.1764	1.5461
1.6859	1.1543
1.5302	1.5606

g) estimate t_0 in f) by the least squares criterion that minimizes the residual sum of squares over (β_1, β_2, t_0).

We know $r_t^e = \beta_1 \mathbb{I}_{\{t < t_0\}} r_M^e + \beta_2 \mathbb{I}_{\{t \geq t_0\}} r_M^e + \epsilon_t$

$$\begin{pmatrix} r_1^e \\ \vdots \\ r_{t0-1}^e \\ r_{t0}^e \\ \vdots \\ r_T^e \end{pmatrix} = \begin{pmatrix} r_{M,1}^e & 0 \\ \vdots & \vdots \\ r_{M,to-1}^e & 0 \\ 0 & r_{M,t0}^e \\ \vdots & \vdots \\ 0 & r_{M,T}^e \end{pmatrix} * \cancel{\text{and } \beta =} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$Y = X$$

From the the least square criterion, we know $\widehat{\beta_t} = (X^T X)^{-1} X^T Y$

Matlab code:

```
clear
fid=fopen('problem3.5.txt');
stockret=textscan(fid,'%s %f %f %f %f %f %f %f %f %f %f','headerlines',
1);
fclose(fid);
fid=fopen('problem3.6.txt');
market=textscan(fid,'%s %f %f','headerlines',1);
fclose(fid);
stocknames={'AAPL', 'ADBE', 'ADP',  'AMD',  'DELL', 'GTW',  'HP',
'IBM',  'MSFT', 'ORCL'};

rm=market{2};    % the monthly return of S&P 500
rf=market{3}/12/100; % the monthly risk-return of treasury bill
for i=1:10
ri=stockret{i+1};
I=ones(length(ri),1);
[B(:,i),BINT(:,:,i), R(:,i), RINT, STATS(i,:)]=regress(ri-rf, [I,rm-rf],
0.05);
subplot(5,2,i),plot(rm-rf,ri-rf,'.')
hold on,
x=-0.1:0.001:0.1;
y=B(1,i)+B(2,i).*x;
plot(x,y,'r')
title(stocknames(i));
sharpe(i)= mean(ri-rf)/std(ri);
treynor(i)= mean(ri-rf)/B(2,i);
end

B
sharpe
treynor

%Using the way of bootstraping pairs of (xi, yi),
M =500; % the number of bootstrap samples
n= length(ri); % the length of original sample
N=n; %% the length of each bootstrap sample
x=zeros(N,M); y=x;
alpha=x;beta=x;coef=zeros(2,M);
for j= 1:10
    ri= stockret{j+1};
    I=ones(N,1);
for i= 1:M
        bootvec = ceil(n.*rand(N,1));
        y(:,i)= ri(bootvec)-rf(bootvec);
        x(:,i)= rm(bootvec)-rf(bootvec);
        [coef(:,i)]= regress(y(:,i),[I, x(:,i)]);
        sharpe(j,i)= mean(y(:,i)-x(:,i))/std(y(:,i));
        treynor(j,i)= mean(y(:,i)-x(:,i))/coef(2,i);
end
alpha(j,:)= coef(1,:);
alphastd(j)= std(coef(1,:));
beta(j,:)=  coef(2,:);
betastd(j)=  std(coef(2,:));
sharpestd(j)= std(sharpe(j,:));
```

```
treynorstd(j)=std(treynor(j,:));
end
alphastd
betastd
sharpestd
treynorstd

a0= B(1,:);
b0= B(2,:);
xt=rm-rf;
xbar=mean(xt);
for i = 1: 10
    yt(i,:)= stockret{i+1}'-rf';
    eps(i,:)=yt(i,:)-a0(i)-b0(i)*xt';
end
n= length(rm);
V0= zeros(10,10);
for j=1: n
V0= V0+ eps(:,j)*eps(:,j)';
end
V0=V0/n;
q=10;
test_stat= (n-q-1)/q*a0*inv(V0)*a0'/(1+n*xbar^2/(sum((xt-xbar).^2)));

 [pc, score, latent, tsquare]= princomp(yt');
cumsum(latent)./sum(latent)
lamda=factoran(yt',6)

t0= find(strcmp(stockret{1},'2/1/2001'));

y1t=yt(:,1:t0-1);
y2t=yt(:,t0:end);
rm1=rm(1:t0-1);
rm2=rm(t0:end);
rf1=rf(1:t0-1);
rf2=rf(t0:end);

for i=1:10
ri1=y1t(i,:)';
I=ones(length(ri1),1);
[B1(:,i),BINT1(:,:,i), R1(:,i), RINT1, STATS1(i,:)]=regress(ri1-rf1,
[I,rm1-rf1], 0.05);
subplot(5,2,i),plot(rm1-rf1,ri1-rf1,'.')
hold on,
x=-0.1:0.001:0.1;
y=B1(1,i)+B1(2,i).*x;
plot(x,y,'r')
title(stocknames(i));
sharpe1(i)= mean(ri1-rf1)/std(ri1);
treynor1(i)= mean(ri1-rf1)/B1(2,i);
end

for i=1:10
ri2=y2t(i,:)';
I=ones(length(ri2),1);
```

```
[B2(:,i),BINT2(:,:,i), R2(:,i), RINT2, STATS2(i,:)]=regress(ri2-rf2,
[I,rm2-rf2], 0.05);
subplot(5,2,i),plot(rm2-rf2,ri2-rf2,'.')
hold on,
x=-0.1:0.001:0.1;
y=B2(1,i)+B2(2,i).*x;
plot(x,y,'g')
title(stocknames(i));
sharpe2(i)= mean(ri2-rf2)/std(ri2);
treynor2(i)= mean(ri2-rf2)/B2(2,i);
end
```