Introduction to Martingale approach for Black-Scholes Models

- Brownian motion
- Ito calculus
- Change of measure
- Martingale representation theorem
- Martingale approach for Black-Scholes model

1. Brownian Motion

- **Definition:** The process $W = (W_t : t \geq 0)$ is a P-Brownian motion if and only if
  - (i) $W_t$ is continuous and $W_0 = 0$;
  - (ii) The value of $W_t$ is distributed, under P, as $N(0,t)$;
  - (iii) The increment $W_{s+t} - W_s$ is distributed as $N(0,t)$, under P, and is independent of $F_s$, the history of what the process did up to time $s$.

- **Odd properties of Brownian motion (BM)**
  - BM is continuous but nowhere differentiable.
  - BM could hit any real value, but, with probability one, it will be back down again to zero.
  - BM is self-similar.
  - BM is also called a Wiener process.
1. Brownian Motion

• Brownian motion with drift

\[ S_t = \sigma W_t + \mu t. \]

– What is the covariance function?

• Geometric Brownian motion (GBM)

– GBM with drift is a model often used for stock prices.
– Assume that \( \mu \) is a drift factor and \( \sigma \) is a noise factor.

\[ X_t = X_0 \exp(\sigma W_t + \mu t). \]

– What is the covariance function?

2. Itô Calculus

• Consider functions of Brownian motion, can we establish certain calculus rules?

• Recall that, in Newtonian differentials

– Differentiable functions can be approximated by piecewise linear functions.

\[ df(X_t) = f'(X_t) dX_t \]

– The old differential tools can NOT be used for Brownian motion. Why?
2. Itô Calculus

• In stochastic differentials,
  – Brownian motions have self-similarity property, in another word, we can NOT approximate Brownian motions (or functions of them) by piecewise linear functions.

  – A stochastic process \( X \) will have a Newtonian term based on \( dt \) and a Brownian term, based on the diffusion term, \( dW_t \).

\[
dX_t = \mu_t dt + \sigma_t dW_t.
\]

2. Itô Calculus

• Itô formula
  – A simple result \( (dW_t)^2 = dt \)
  – For a smooth function \( f \), consider a Taylor expansion

\[
df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)(dW_t)^2 + \frac{1}{3!}f^{(3)}(W_t)(dW_t)^3 + \cdots
\]

Therefore,

\[
df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt.
\]
2. Itô Calculus

- If $X$ is a stochastic process,
  \[ dX_t = \mu_t dt + \sigma_t dW_t, \]
  and $f$ is a deterministic twice continuously differentiable function, then $f(X_t)$ is also a stochastic process and is given by
  \[
  df(X_t) = f'(X_t) dX_t + \frac{1}{2} f''(X_t)(dX_t)^2
  \]
  \[ = \sigma_t f'(X_t) dW_t + [\mu_t f'(X_t) + \frac{1}{2} \sigma_t^2 f''(X_t)] dt \]

2. Itô Calculus

- Example
  1. Let $X_t = W_t^2$, then $dX_t = 2W_t dW_t + dt$
  2. Exponential Brownian motion: $X_t = e^{\sigma W_t + \mu t}$, then
     \[ dX_t = X_t [\sigma dW_t + (\mu + \frac{1}{2} \sigma^2) dt] \]
  3. If $dX_t = X_t(\sigma dW_t + \mu dt)$, then what is $X_t$?
     Let $Y_t = ln(X_t)$, we then have $dY_t = (\sigma dW_t + \mu dt) - \frac{1}{2} \sigma^2 dt$
     \[ X_t = X_0 e^{(\mu - \frac{1}{2} \sigma^2)t + \sigma W_t}. \]
3. Change of Measure

- We have used Brownian motions to describe the stock price movement, and also got basic tools for differentials of Brownian motion.

- Note that a Brownian motion $W_t$ is referred to some measure $P$, or we should call $W_t$ a \textit{P-Brownian motion}.

- However, we have no idea how $W_t$ or $X_t$ changes as the measure changes.

3. Change of Measure

- Equivalent measures
  Two measures $P$ and $Q$ are equivalent if they operate on the same sample space and agree on what is possible. If $A$ is any event in the sample space, $P(A) > 0 \iff Q(A) > 0$

- $dQ/dP$ and $dP/dQ$ are only well-defined if measures $P$ and $Q$ are equivalent.
3. Change of Measure

- **Radon-Nikodym process**
  
  Let $\zeta_t$ be the Radon-Nikodym derivative taken up to the horizon $t$, i.e.,
  \[
  \zeta_t = E_P \left( \frac{dQ}{dP} | \mathcal{F}_t \right),
  \]
  
  then
  \[
  E_Q(X_t) = E_P(\zeta_t X_t) = E_P \left( \frac{dQ}{dP} X_t \right) \]
  \[
  E_Q(X_t | \mathcal{F}_s) = \zeta_s^{-1} E_P(\zeta_t X_t | \mathcal{F}_s), \quad s \leq t \leq T,
  \]

- **Cameron-Martin-Girsanov theorem**

  If $W_t$ is a $P$-Brownian motion and $\gamma_t$ is an $\mathcal{F}$-previsible process satisfying the boundeness condition
  \[
  E_P \exp \left[ \frac{1}{2} \int_0^T \gamma_t^2 dt \right] < \infty,
  \]
  
  then there exists a measure $Q$ such that
  
  (i) $Q$ is equivalent to $P$;
  (ii) $\frac{dQ}{dP} = \exp(-\int_0^T \gamma_t dW_t - \frac{1}{2} \int_0^T \gamma_t^2 dt)$
  (iii) $\tilde{W}_t = W_t + \int_0^t \gamma_s ds$ is a $Q$-Brownian motion.
3. Change of Measure

• Cameron-Martin-Girsanov converse

If \( W_t \) is a P-Brownian motion and \( Q \) is a measure equivalent to \( P \), then there exists an \( \mathcal{F} \)-previsible process \( \gamma_t \) such that

\[
\tilde{W}_t = W_t + \int_0^t \gamma_s ds
\]

is a \( Q \)-Brownian motion. That is, \( W_t \) plus drift \( \gamma_t \) is \( Q \)-Brownian motion. Additionally the Radon-Nikodym derivative of \( Q \) with respect to \( P \) at time \( T \) is

\[
\exp\left(-\int_0^T \gamma_t dW_t - \frac{1}{2} \int_0^T \gamma_t^2 dt\right).
\]

3. Change of Measure

• C-M-G and stochastic differentials

Suppose that \( dX_t = \sigma_t dW_t + \mu_t dt \), where \( W \) is a P-Brownian motion. Is there a measure \( Q \) such that the drift of \( X \) under \( Q \) is \( \nu_t dt \) instead of \( \mu_t dt \)?

\[
\begin{align*}
dX_t &= \sigma_t dW_t + \mu_t dt \\
     &= \sigma_t \left( dW_t + \frac{\mu_t - \nu_t}{\sigma_t} dt \right) + \nu_t dt \\
     &= \sigma_t d\tilde{W}_t + \nu_t dt.
\end{align*}
\]
4. Martingale Representation Theorem

• Martingale

A stochastic process $M_t$ is a martingale with respect to a measure $P$ if and only if

(i) $E_P(|M_t|) < \infty$, for all $t$

(ii) $E_P(M_t|\mathcal{F}_s) = M_s$, for all $s \leq t$.

For a martingale, the expected future value conditional on its present value and past history is equal to its present value.

4. Martingale Representation Theorem

• Martingale examples

(1) $P$-Brownian motion $W_t$ is a $P$-martingale.

\[ E_P(W_t|\mathcal{F}_s) = W_s. \]

(2) $N_t = E_P(X|\mathcal{F}_t)$ is a $P$-martingale, where $X$ depends only on events up to time $T$. 
4. Martingale Representation Theorem

Suppose that $M_t$ is a Q-martingale process, whose volatility $\sigma_t$ satisfies the additional condition that it is always non-zero. Then if $N_t$ is any other Q-martingale, there exists an $\mathcal{F}$-previsible process $\phi$ such that

$$\int_0^T \phi_t^2 \sigma_t^2 dt < \infty$$

with probability one, and $N$ can be written as

$$N_t = N_0 + \int_0^t \phi_s dM_s,$$

where $\phi$ is unique.

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4. Martingale Representation Theorem

- **Martingales and driftlessness**
  If $X$ is a stochastic process with volatility $\sigma_t$, that is

$$dX_t = \sigma_t dW_t + \mu_t dt$$

which satisfies the technical condition

$$E\left( \left( \int_0^T \sigma_s^2 ds \right)^{\frac{1}{2}} \right) < \infty,$$

then $X$ is a martingale $\iff$ $X$ is driftless ($\mu_t \equiv 0$).

- **Exponential martingales**
  If $dX_t = \sigma_t X_t dW_t$, for some $\mathcal{F}$-previsible process $\sigma_t$, then

$$E\left( \exp\left( \frac{1}{2} \int_0^T \sigma_s^2 ds \right) \right) < \infty \implies X \text{ is a martingale.}$$
5. Martingale approach for Black-Scholes model

- Black-Scholes model is the basic building blocks of derivatives theory.

- In 1970s, Fisher Black, Myron Scholes and Robert Merton made a major breakthrough in the pricing of stock options – they develop the Black-Scholes (or Black-Scholes-Merton) model.

- Merton and Scholes received the Nobel prize in 1997.

5. Martingale approach for Black-Scholes model

1. Take a stock model

2. Use the Cameron-Martin-Girsanov theorem to change it into a martingale.

3. Use the martingale representation theorem to create a replicating strategy for each claim.
5. Martingale approach for Black-Scholes model

Model assumptions:
– The stock price and bond price follow

\[ S_t = S_0 \exp(\mu t + \sigma W_t), \quad B_t = \exp(rt), \]

where \( r \) is the riskless interest rate, \( \sigma \) is the stock volatility and \( \mu \) is the stock drift (for simplicity, we assume that \( r, \mu \) and \( \sigma \) are deterministic).
– There are no transaction costs.
– Both instruments are freely and instantaneously tradable either long or short at the price quoted.

5. Martingale approach for Black-Scholes model

Three steps to replication

• Find a measure \( Q \) under which \( S_t \) is a martingale.

• Form the process

\[ E_t = E_Q(X|\mathcal{F}_t). \]

• Find a previsible process \( \phi_t \), such that

\[ dE_t = \phi_tdS_t. \]
5. Martingale Approach – Step One

1. We first find a stochastic differential equation for $S_t$.

$$ S_t = \exp(\sigma W_t + \mu t) $$

$$ \implies dS_t = \sigma S_t dW_t + (\mu + \frac{1}{2} \sigma^2) S_t dt. $$

2. In order for $S_t$ to be a martingale, we need to remove the drift. Let $\gamma_t \equiv (\mu + \frac{1}{2} \sigma^2)/\sigma$, then based on the C-M-G theorem, there is a measure $Q$ such that $\tilde{W}_t = W_t + \gamma t$ is a $Q$-Brownian motion. Therefore, we have

$$ dS_t = \sigma S_t d\tilde{W}_t, $$

and $S_t$ become a $Q$-martingale.

5. Martingale Approach – Step Two

For an arbitrary claim $X$ (e.g., it is $\max(S_T - K, 0)$ for an European call option),

$$ E_t = E_Q(X|F_t) $$

is a $Q$-martingale.
5. Martingale Approach – Step Three

As $E_t$ and $S_t$ are both $Q$-martingales, based on the martingale representation theorem, there exists a previsible process $\phi_t$ such that

$$E_t = E_Q(X|F_t) = E_Q(X) + \int_0^t \phi_s dS_s,$$

or $dE_t = \phi_t dS_t$.

5. Martingale Approach – Summary

Suppose we have a Black-Scholes model for a continuously tradable stock and bond, which are represented by

$$S_t = S_0 \exp(\mu t + \sigma W_t), \quad B_t = \exp(rt),$$

respectively. Then

(1) all integrable claims $X_T$ have associated replicating strategies $(\phi_t, \psi_t)$.

(2) The arbitrage price of such a claim $X$ is given by

$$V_t = B_tE_Q(B_T^{-1}X_T|\mathcal{F}_t) = e^{-(T-t)}E_Q(X_T|\mathcal{F}_t),$$

where $Q$ is the martingale measure for the discounted stock $B_t^{-1}S_t$. 
5. Martingale Approach – European Options

The claim \( X_T \) is \((S_T - K)^+ := \max(S_T - K, 0)\). Then,

\[ V_0 = e^{-rT} E_Q((S_T - K)^+). \]

**Question:** How to evaluate this function?

— We first find the marginal distribution of \( S_T \) under \( Q \):

\[
d(\log S_t) = \sigma d\bar{W}_t + (r - \frac{1}{2}\sigma^2)dt
\]

\[ \Rightarrow \quad S_t = S_0 \exp(\sigma \bar{W}_t + (r - \frac{1}{2}\sigma^2)t) \]

5. Martingale Approach

**Question:** How to evaluate this function?

\[ V_0 = e^{-rT} E_Q((S_T - K)^+). \]

Let \( Z \sim N(-\frac{1}{2}\sigma^2 T, \sigma^2 T) \), then \( S_T = S_0 \exp(Z + rT) \), and

\[
V_0 = e^{-rT} E((S_0 e^{Z+rT} - K)^+)
\]

\[
= \frac{1}{\sqrt{2\pi \sigma^2 T}} \int_{\log(S_0)}^{\infty} (S_0 e^x - ke^{-rT}) \exp\left(-\frac{(x + \frac{1}{2}\sigma^2 T)^2}{2\sigma^2 T}\right) dx
\]

\[ \Rightarrow \quad S_0 \Phi\left(\frac{\log \frac{S_0}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma \sqrt{T}}\right) - Ke^{-rT} \Phi\left(\frac{\log \frac{S_0}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma \sqrt{T}}\right). \]

\[ d_1 \quad d_2 \]
5. Martingale Approach – Other Options

1. The put option has a payoff $X_T = \max(K - S, 0)$, hence its value at time 0 is

$$-S\Phi(-d_1) + Ke^{-rT}\Phi(-d_2).$$

2. For a binary call with $Payoff(S) = H(S - K)$, where $H$ is the Heaviside function, the value of the option at time 0 is

$$e^{-rT}\Phi(d_2).$$

3. For a binary put with $Payoff(S) = H(K - S)$, the option has a value of

$$e^{-rT}(1 - \Phi(d_2)).$$