AMS 502 Final Examination

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1. Prove $u = F(x + at) + G(x - at)$ satisfies the wave equation $u_{tt} = a^2 u_{xx}$, where $a > 0$ is a constant, and F, G are sufficiently smooth functions.

2. Find Fourier transformation: $f(x) = Ae^{-|\mathbf{x}|^2}, \mathbf{x} \in \mathbb{R}^n$, where $A > 0$ is constant.

3. Suppose Ω is a smooth, bounded domain in $\mathbb{R}^n$ and $u, v \in C^2(\bar{\Omega})$. Prove the following Green's identities, where dS denotes the surface measure on $\partial\Omega$ and ν is the exterior unit normal on $\partial\Omega$.

$$\int_{\partial\Omega} v \frac{\partial u}{\partial \nu} dS = \int_{\Omega} (v \triangle u + \nabla u \nabla v) dx,$$
$$\int_{\partial\Omega} (v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu}) dS = \int_{\Omega} (v \triangle u - u \triangle v) dx.$$

4. Find the solution $u(x, t)$ to the following heat equation, where g is continuous and bounded for $x \geq 0$ with $g(0) = 0$.

$$\begin{aligned} u_t &= u_{xx}, \quad && x > 0, t > 0, \\ u(x, 0) &= g(x), && x > 0, \\ u(0, t) &= 0, && t > 0. \end{aligned}$$

5. Let $\Omega = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < 1\} = \{(r, \theta) : 0 \leq r < 1, \quad 0 \leq \theta < 2\pi\}$. Use separation of variables (r, θ) to solve the Dirichlet problem

$$\begin{aligned} \triangle u &= 0, && (r, \theta) \in \Omega, \\ u(1, \theta) &= g(\theta), && 0 \leq \theta < 2\pi. \end{aligned}$$

6. Find Riemann Invariants for the following shallow-water waves equations, where $g > 0$ is a constant.

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} &= 0, \\ \frac{\partial (hu)}{\partial t} + \frac{\partial (hu^2 + \frac{1}{2} g h^2)}{\partial x} &= 0. \end{aligned}$$

7. Find the solution $u(x, y)$ to

$$\begin{aligned} u_x^2 + u_y^2 - u &= 0, \\ u(x, 0) &= \frac{x^2}{8}. \end{aligned}$$

8. Find the solution to

$$\begin{aligned} u_t + \frac{\partial (\frac{u^2}{2})}{\partial x} &= 0, \\ u|_{t=0} &= \begin{cases} 0, & x < 0, \\ 1, & x \in [0, 1], \\ -1, & x > 1. \end{cases} \end{aligned}$$