A Multiphase Flow Model for the Unstable Mixing of Layered Incompressible Materials

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Abstract

In this paper, a model for the unstable mixing of \( n \) parallel or concentric incompressible fluid layers is proposed. The approach to constructing this model is pairwise, based on a known two incompressible fluid mixing model. The problem complexity increases significantly in going from two to three fluids, but the increase in complexity is relatively small thereafter. We present a detailed study of the \( n = 3 \) problem, which displays all of the difficult modeling issues applicable to arbitrary \( n \geq 3 \) while still being reasonably tractable.
1 Introduction

The problem of hydrodynamic instability of fluid interfaces has received much attention in recent years due to its strong effect on the implosion of inertial confinement fusion targets [1]. The problem of instabilities occurring at two or more interfaces in a multilayered system has not, however, been studied extensively. Over the past two decades, some studies on multilayer systems have been conducted [2, 3, 5, 6, 7, 8], including the recent works [9, 10]. In this paper, we analyze this problem using an incompressible multiphase flow model. The physical picture is an $n$-layered fluid flow, where each layer contains a different material and some or all of the $n - 1$ interfaces separating the distinct materials are unstable due to an external acceleration or impulsive load, such as occurs in inertial confinement fusion. The configuration of the layers is either planar, with the density stratification aligned along the $z$-axis, or cylindrical/spherical, with the density stratification aligned along the radial direction. The direction of the externally imposed force is vertical, i.e., parallel to the density gradient, so that the subsequent mixing layer formation is statistically uniform in any horizontal plane or shell.

Recent papers [11, 12, 13, 14] proposed an interpolation scheme for the interfacial closures for two fluid mixing layers. These closures led to analytic solutions for the mixture fractions and velocities in mixing layers formed from incompressible fluids. Model predictions were in agreement with all available experimental data, although this data did not provide a stringent test of the modeling assumptions. In the $n$-fluid mixing problem, it remains true that all interfaces separate exactly two fluids, so that by considering the interaction of fluid $j$ with fluid $k$, $j \neq k$, across the interface separating these fluids, we construct interface closures for $n$ materials as a weighted sum of pairwise contributions.

2 Averaged Equations

Let the $n$ materials be indexed by a subscript $k$. Following previous analyses of two-phase flow [15, 16], we define the phase indicator function $X_k$ for $k = 1, \ldots, n$, a function of
spatial position $\vec{x}$ and time $t$,\[ X_k(\vec{x},t) = \begin{cases} 1 & \text{if } \vec{x} \text{ is in fluid } k \text{ at time } t \\ 0 & \text{otherwise} \end{cases} \quad k = 1, \ldots, n.\]

Denoting ensemble averages by $\langle \cdot \rangle$, the quantity $\langle X_k \rangle (\vec{x},t)$ is the probability that in a randomly selected flow realization the space-time position $(\vec{x},t)$ is occupied by fluid $k$. We refer to $\langle X_k \rangle$ as the mixture volume fraction of fluid $k$ and denote it $\beta_k$.

Assuming that the materials do not exchange mass, it follows that $X_k$ is invariant along fluid particle trajectories. In other words, the material derivative $\frac{\partial X_k}{\partial t} + \vec{v} \cdot \nabla X_k$ equals zero [15], where $\vec{v}$ is the flow velocity. Ensemble averaging this equation and imposing the slab or shell symmetry described above, we obtain
\[ \frac{\partial \beta_k}{\partial t} + v_k^* \frac{\partial \beta_k}{\partial h} = 0, \tag{1} \]
where $v_k^*$ is an effective velocity for interfaces adjacent to material $k$,
\[ v_k^* \equiv \frac{\langle \vec{v} \cdot \nabla X_k \rangle}{\beta_k}, \tag{2} \]
and $h$ is the spatial coordinate in the vertical or longitudinal direction, i.e. $h = z$ for a planar configuration and $h = r$ for a cylindrical/spherical configuration.

Assuming incompressible fluids, we multiply the microscopic continuity condition $\nabla \cdot \vec{v} = 0$ by $X_k$, average, and apply the product rule for derivatives, so that
\[ 0 = \langle X_k \nabla \cdot \vec{v} \rangle = \langle \nabla \cdot (X_k \vec{v}) \rangle - \langle \vec{v} \cdot \nabla X_k \rangle. \]
Commuting the ensemble average with the gradient operator, imposing slab or shell symmetry and applying the definitions of the mixture volume fraction $\beta_k$ and effective interface velocity $v_k^*$, we obtain a macroscopic equation for fluid $k$,
\[ \frac{\partial \beta_k v_k}{\partial h} + s \frac{\beta_k v_k}{h} = v_k^* \frac{\partial \beta_k}{\partial h}, \tag{3} \]
where $v_k \equiv \langle X_kv \rangle / \beta_k$ is the average velocity of fluid $k$ and $s = 0, 1, \text{ and } 2$ for planar, cylindrical, and spherical geometry, respectively.
Summing Eq.(3) over all materials, applying the definition of \(v_k^*\), commuting the sum with the ensemble average, and applying the volume-filling constraint \(\sum_{k=1}^{n} X_k = 1\), the right hand of the equation becomes

\[
\sum_{k=1}^{n} v_k^* \frac{\partial \beta_k}{\partial h} = 0. \tag{4}
\]

When there are only two fluids, this condition is equivalent to \(v_1^* = v_2^* \equiv v^*\), thus the two-phase flow model involves just a single interface velocity [16]. This case has been analyzed extensively [11, 12, 14], using a model for \(v^*\), proposed in [13], that involves only \(t, \beta_k, \) and \(v_k\). On the other side, the left hand of the equation gives

\[
\frac{\partial}{\partial h} \sum_{k=1}^{n} \beta_k v_k = -\frac{s}{h} \sum_{k=1}^{n} \beta_k v_k. \tag{5}
\]

For any value of \(h\) satisfying Eq. (5), it requires

\[
\sum_{k=1}^{n} \beta_k v_k \equiv 0. \tag{6}
\]

The dynamic model equations of motion for \(n\) materials are the following

\[
\frac{\partial \beta_k}{\partial t} = -v_k^* \frac{\partial \beta_k}{\partial h}, \tag{7}
\]

\[
\frac{\partial \rho_k}{\partial t} + \frac{\partial (\rho_k v_k)}{\partial h} = -\frac{s}{h} \rho_k v_k^2 + \frac{\rho_k (v_k^* - v_k)}{\beta_k} \frac{\partial \beta_k}{\partial h}, \tag{8}
\]

\[
\frac{\partial (\rho_k v_k^2)}{\partial t} + \frac{\partial (\rho_k v_k^2 + p_k)}{\partial h} = -\frac{s}{h} \rho_k v_k^2 + \rho_k g(t) + \frac{1}{\beta_k} [\rho_k v_k (v_k^* - v_k) + p_k - p] \frac{\partial \beta_k}{\partial h}, \tag{9}
\]

\[
\frac{\partial (\rho_k e_k)}{\partial t} + \frac{\partial (\rho_k e_k v_k + p_k v_k)}{\partial h} = -\frac{s}{h} (\rho_k e_k v_k + p_k v_k) + \rho_k v_k g(t)
+ \frac{1}{\beta_k} [\rho_k e_k (v_k^* - v_k) + (pv)_k^* v_k - 2p_k v_k] \frac{\partial \beta_k}{\partial h}, \tag{10}
\]

where the volume fraction \(\beta_j\) satisfies

\[
\sum_{j=1}^{n} \beta_j = 1. \tag{11}
\]

In multilayer systems, a single fluid EOS holds within each fluid. The quantities \(v_k^*, p_k^*,\) and \((pv)_k^*\) represent averages of microscopic quantities (products of primitive variables).
that need to be modeled. Specifically, \( q^*_k \) denotes a weighted average of the fluid quantity \( q \), conditioned on evaluation at the interface adjacent to the \( k \) material. Surface tension is neglected in this model, so that \( p^*_k \) and \( (p\nu)^*_k \) are well-defined quantities.

Each layer has both a minimum and a maximum penetration as demonstrated in Fig. 1. These “edges” follow certain trajectories which are additional degrees of freedom in the multiphase flow model. Let \( H^\pm_k(t) \) denote the largest/smallest (\(+/−\)) height at which fluid \( k \) is present at time \( t \), and denote the corresponding velocities \( V^\pm_k(t) = \frac{dH^\pm_k}{dt} \). Of course, the maximum penetration of the outermost layer and the minimum penetration of the innermost layer are not used, except perhaps to define the boundaries of the fluid domain.

The as yet unspecified trajectories of the fluid layer edges serve as boundary conditions for the solution of the system of differential equations (1) and (3). With the assumptions of incompressible flow and a closure for \( v^*_k \) that does not depend on pressure, the momentum equations do not play a direct role in determining the solutions for the mixture fractions and velocities. The coupling of the interface and continuity equations with the momentum equations can only occur at the layer edges, for example,
if a model closure depends on pressure.

In the following section, we review the two fluid interface closure and then use it to develop an interface closure for an \( n \)-layered mixing problem.

## 3 Closure for Effective Interface Quantities

### 3.1 \( n = 2 \)

We first consider a single interface separating two fluids which is hydrodynamically unstable. Layer 1 is initially below layer 2, so that the mixing layer which develops is confined to the spatial region \( H_2^- (t) \leq h \leq H_1^+ (t) \). As there are no intermediate layers in this case, we may drop the \( \pm \) superscripts without possibility of confusion, and regard \( H_k(t) \) as the tip of the furthest penetrating portion of fluid \( k \), \( i.e. \)

\[
\lim_{h \to H_2^-} \beta_2 = 0, \quad \lim_{h \to H_1^+} \beta_1 = 0.
\]

Consequently we have the following boundary conditions on the bulk and interface velocities,

\[
\lim_{h \to H_2^-} v_2 = \lim_{h \to H_2^-} v^* = V_2(t), \quad \lim_{h \to H_1^+} v_1 = \lim_{h \to H_1^+} v^* = V_1(t),
\]

where \( V_k \equiv \frac{dH_k}{dt} \) is the velocity of edge \( k \).

The model for \( v^* \), explained in [13, 12], has the form

\[
v^* = \mu_k^v (\beta_k, t) v_{k'} + \mu_{k'}^v (\beta_{k'}, t) v_k,
\]

(12)

where \( k' = 3 - k \). Assuming that \( v^* \) is a function of \( v, \beta \) and \( t \) alone, the linear expression (12) can be derived from fundamental principles (scale covariance and frame invariance) in planar geometry, see [12]. The required frame invariance of \( v^* \) (in planar geometry) is equivalent to the constraint \( \mu_k^v + \mu_{k'}^v = 1 \). An alternate derivation can be based on manipulation of the unclosed equations (8-10). These equations can be rewritten to yield an expression of the form (12) and moreover the convex coefficients \( \mu_k^v \) are fractional linear in \( \beta \) [4]. Further algebraic constraints on the coefficients \( \mu_k^v \) are provided by the boundary conditions on \( v^* \). If \( \mu_k^v \) is assumed to be a fractional linear function of \( \beta_k \), then
the only possibility is
\[ \mu_k^v(t, \beta_k) = \frac{\beta_k}{\beta_k + \sigma_k^v(t)\beta_k'}, \] (13)
where now frame invariance is reduced to the constraint \( \sigma_k^v\sigma_k^{v'} = 1 \). Thus we close the \( v^* \) equation by introducing a a single undetermined function of time, \( \sigma_k^v \). In curved geometry, where the longitudinal component of \( \vec{v}_k \) is not required to be frame invariant, we retain the constraint \( \mu_k^v + \mu_k^{v'} = 1 \) as an additional assumption. The above derivation also offers an interpretation of \( \sigma_k^v \) as the ratio of mixed fluid volume creation for the two fluids, and thus in the incompressible case, we have the constraint \( \sigma_k^v = V_{k'}/V_k \).

Similarly, the model for \( p^* \) in two fluids system, explained in [13], has the form
\[ p^* = \mu_k^p(\beta_k, t)p_{k'} + \mu_k^{p'}(\beta_k', t)p_k, \] (14)
and
\[ \mu_k^p(t, \beta_k) = \frac{\beta_k}{\beta_k + \sigma_k^p(t)\beta_k'}, \] (15)
where the parameter \( \sigma_k^p \) can be determined as a function of the fluid densities, for example, \( \sigma_k^p = \rho_{k'}/\rho_k \). As with \( v^* \), the convex interpolation formula (14) and the fractional linear form (15) are exact consequences of the unclosed equations [4], and closure is provided by the explicit choice assumed for \( \sigma_k^p \). We can interprete \( \sigma_k^p \) as the ratio of the specific forces accelerating the two fluids.

3.2 \( n \geq 2 \)

We now propose a scheme to close the interface and continuity equations for arbitrary \( n \geq 2 \), up to specification of the edge trajectories \( H_{k}^{\pm}(t) \), by specification of \( v_k^* \) as a function of volume fractions \( \beta_k \), bulk velocities \( v_k \), and time \( t \).

The effective interface velocity for fluid \( k \) can be decomposed into a sum of contributions from each type of interface adjoining fluid \( k \). Let \( v_{kj} \) be the effective velocity, at given height \( h \) and time \( t \), of all interfaces separating fluids \( k \) and \( j \), \( k \neq j \). At this space-time point, these interfaces are present as some proportion of all of the interfaces...
adjoining fluid $k$. Call this proportion $\Psi_{jk}$. It follows that

$$v_k^* = \sum_{\substack{j=1 \atop j \neq k}}^n \Psi_{jk} v_{kj}.$$  \hfill (16)

As $v_{kj}$ represents an interaction between two fluids, it is modeled as a two-fluid effective interface velocity,

$$v_{kj} = \mu_{kj}^v v_k + \mu_{jk}^v v_j.$$  \hfill (17)

In the limit as fluid $j$ vanishes, there is only a single point in the ensemble contributing to $v_{kj}$, so that

$$\lim_{\beta_j \to 0} v_{kj} = v_j.$$  

Here, and later in this section, we impose frame invariance to derive an algebraic constraint on the model. As mentioned above, we retain this constraint in the case of curved geometry as an additional modeling assumption. The two constraints of frame invariance and vanishing $\beta_j$ imposed on an assumed fractional linear form for $\mu_{kj}^v$ imply the model

$$\mu_{kj}^v = \frac{\beta_j}{\beta_j + \sigma_{kj}^v \beta_k}, \quad \mu_{jk}^v = 1 - \mu_{kj}^v.$$  \hfill (18)

with the parameter $\sigma_{kj}^v$ to be specified.

The quantity $\Psi_{jk}$ is the relative contribution (or probability) of $j$-$k$ interfaces to the effective velocity of all interfaces adjoining fluid $k$, thus it satisfies the following constraints,

$$\lim_{\beta_j \to 0} \Psi_{jk} = 0, \quad \sum_{j \neq k} \Psi_{jk} = 1.$$  

We assume that $\Psi_{jk}$ is proportional to the relative amount of fluid $j$ in the mixture, i.e. $\beta_j$. This simplifying assumption does not distinguish between a large number of small chunks of fluid $j$ or a small number of large chunks of fluid $j$; both may have similar $\beta_j$, but the latter situation implies a significantly smaller contribution to $v_k^*$. It follows from the assumption $\Psi_{jk} \propto \beta_j$ that

$$\Psi_{jk} = \frac{\beta_j}{1 - \beta_k}.$$  \hfill (19)

This choice of $\Psi_{jk}$, combined with the frame invariance of the $v_{kj}$, guarantees that the model (16) is also frame invariant.
At a fluid layer edge, \( v_k^* \) must coincide with \( v_k \), which in turn coincides with the velocity of the edge. We now confirm that this boundary condition is satisfied by the proposed model for \( v_k^* \), Eq. (16). Setting \( \beta_k = 0 \) in the expressions above, we have \( \Psi_{jk} = \beta_j \) and \( v_{kj} = v_k \), so that

\[
v_k^* = \sum_{j \neq k} \beta_j v_k = v_k \sum_{j \neq k} \beta_j = v_k.
\]

The fact that \( v_k^* \) is nonzero where it is undefined, namely in a region excluding fluid \( k \), is irrelevant because \( v_k^* \) appears in the continuity equation multiplied by \( \frac{\partial \beta_k}{\partial h} \), which equals zero in such regions.

With the proposed models for \( \Psi_{jk} \) and \( v_{kj} \), our ansatz (16) for the interfacial velocity \( v_k^* \) becomes

\[
v_k^* = \sum_{j \neq k} \frac{\beta_j}{(1 - \beta_k)(\beta_j + \sigma_{kj}^v \beta_k)} \left( \beta_j v_k + \sigma_{kj}^v \beta_k \right).
\]

For \( k = 1, 2, \ldots, n \), this can be written in matrix form

\[
\begin{pmatrix}
v_1^* \\
\vdots \\
v_n^*
\end{pmatrix} =
\begin{pmatrix}
A_{11} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & A_{nn}
\end{pmatrix}
\begin{pmatrix}
v_1 \\
\vdots \\
v_n
\end{pmatrix} +
\begin{pmatrix}
B_{11} & \cdots & B_{1n} \\
\vdots & \ddots & \vdots \\
B_{n1} & \cdots & B_{nn}
\end{pmatrix}
\begin{pmatrix}
v_1 \\
\vdots \\
v_n
\end{pmatrix},
\]

where

\[
A_{kk} = \delta_{kj} \sum_{l \neq k} \frac{\beta^2_l}{(1 - \beta_k)(\beta_l + \sigma_{kl}^v \beta_k)}
\]

represents the isotropic component that describes the contributions from the same materials and

\[
B_{kj} = \frac{\sigma_{kj}^v \beta_j \beta_k}{(1 - \beta_k)(\beta_j + \sigma_{kj}^v \beta_k)}
\]

is the nonisotropic component which describes the contributions from the different materials.

For incompressible flow, the continuity equations (3) together with Eq. (6) give \((n-1)\) independent differential equations

\[
\beta_k \frac{\partial v_k}{\partial h} = -\frac{s}{h} \beta_k v_k + (v_k^* - v_k) \frac{\partial \beta_k}{\partial h}, \quad k = 1, 2, \cdots n - 1
\]
for the \((n - 1)\) independent variables \(\{v_k\}\), \(k = 1, 2, \ldots, n - 1\). The variables of the \(n\)th material are given by

\[
\beta_n = 1 - \sum_{j=1}^{n-1} \beta_j, \quad \beta_n v_n = - \sum_{j=1}^{n-1} \beta_j v_j.
\]

Substituting \(v_k^*\) into above equations, the \(n - 1\) differential equations can be further simplified to give

\[
\frac{\partial v_k}{\partial h} = -\frac{s}{h} v_k + \frac{1}{1 - \beta_k} \sum_{j \neq k}^{n} \frac{\sigma_{kj}^v \beta_j (v_j - v_k)}{\sigma_{kj}^v \beta_k + \beta_j} \frac{\partial \beta_k}{\partial h}, \quad k = 1, 2, \ldots, n,
\]

where

\[
v_k^* - v_k = \frac{1}{1 - \beta_k} \sum_{j \neq k} \left[ \left( \frac{\beta_j^2}{\beta_j + \sigma_{kj}^v \beta_k} - \beta_j \right) v_k + \frac{\sigma_{kj}^v \beta_j \beta_k}{\beta_j + \sigma_{kj}^v \beta_k} v_j \right] = \frac{\beta_k}{1 - \beta_k} \sum_{j \neq k} \frac{\sigma_{kj}^v \beta_j (v_j - v_k)}{\beta_j + \sigma_{kj}^v \beta_k}.
\]

Similarly, the effective pressure for the interface adjacent to the \(k\)th material can be expressed as

\[
p^* = \sum_{j \neq k}^{n} \psi_{jk} p_{kj},
\]

where

\[
p_{kj} = \mu_{kj}^p p_k + \mu_{jk}^p p_j,
\]

\[
\mu_{kj}^p = \frac{\beta_j \rho_j}{\beta_j \rho_j + \beta_k \rho_k}, \quad \mu_{kj}^p + \mu_{jk}^p = 1.
\]

With these closures, the model equations are completed and can be solved numerically.

\section{Boundary Conditions}

In this section, we complete the closure of the \(n\)-material mixing model by proposing a model for the edge trajectories \(H_k^{\pm}(t)\).

In the case of two fluids, the penetration of the light material into the heavy material resembles an array of bubbles, while the penetration of the heavy into light resembles an array of mushroom-capped spikes. Of course, the distinction between “spike” and
“bubble” disappears as \( A \rightarrow 0 \). Assume \( \rho_1 < \rho_2 \). Then the position of the tip of the furthest penetrating bubble defines the trajectory of the \( \beta_1 = 0 \) characteristic, which is the upper edge of the mixing layer. Similarly, the tip of the furthest penetrating spike defines the trajectory of the \( \beta_2 = 0 \) characteristic, which is the lower edge of the mixing layer. These trajectories are additional degrees of freedom in the two-phase flow model, i.e. they are not determined by the closure assumptions made thus far.

The case \( n \geq 3 \) is drastically different from the \( n = 2 \) case due to the presence of intermediate fluid layers. Depending on its initial thickness, an intermediate layer may not supply enough fluid to form the coherent structures that are routinely observed in the two-fluid case. Thus a model based on the behavior of these coherent structures may not be applicable. A further complication is the possibility that one or both boundaries of the layer are stable. A stable interface will not distort until it is penetrated by a spike or bubble formed at a neighboring interface.

At all times, and for any \( n \), the penetration of an individual material has a finite extent. The corresponding trajectory must be added to the multiphase model as a boundary condition. Each intermediate fluid, i.e. one of fluids 2 through \( n - 1 \), is terminated on both sides by a material interface, thus it penetrates neighboring layers in two directions. We have already introduced \( H_k^\pm(t) \) as the trajectory of the upper (+) and lower (−) edge of layer \( k; \ k = 2, \ldots, n - 1 \). Note that any individual fluid can be pictured as a layer provided that it continues to possess an unmixed portion, even though it is bound by two highly distorted interfaces. More generally, \( H_k^\pm(t) \) is the (largest, smallest) height at which fluid \( k \) is present. We order the layers starting with the lowest, so that in curved geometry fluid 1 is the inner core, fluid \( n \) the outer core. Then fluid 1 penetrates outward or upward with trajectory \( H_1^+(t) \equiv H_1(t) \) and fluid \( n \) penetrates inward or downward with trajectory \( H_n^-(t) \equiv H_n(t) \). We add the following constraints to the system of multiphase flow equations,

\[
\beta_k \left( H_k^\pm(t), t \right) = 0, \quad k = 2, \ldots, n - 1, \\
\beta_1 (H_1(t), t) = \beta_n ((H_n(t), t) = 0.
\]
The remaining trajectories $H^+_1$ and $H^-_n$ have no meaning in curved geometry, but they can be thought of as constants defining the upper and lower walls of the spatial domain in planar geometry.

For shock and acceleration driven mixing of two fluids in planar geometry, the recent experiments of Dimonte and Schneider [17] provide data to calibrate a buoyancy/drag model [18, 19] for the spike/bubble tips of the form

$$\frac{dV}{dt} = Ag - \frac{C \rho'}{\rho + \rho'} V(t) \left| \frac{V(t)}{H(t) - H(0)} \right|,$$

where the drag coefficient $C$ is structure and Atwood number dependent, $H$ is the edge trajectory, and the edge velocity $V = \frac{dH}{dt}$ is relative to an ambient background. The underlying assumption of this model is the formation and interpenetration of bubble and spikes. Thus it should be applicable to two-fluid mixing in any geometry provided that suitable values of the coefficients are found. In the presence of shearing, the distortion of the bubbles and spikes may be significant enough to require the addition of other forces to the model (24) or a different form for the drag force. The two initial blocks of fluid are generally large enough to feed the formation of bubbles and spikes; however, one possible exception is a spherical or cylindrical interface problem with a very small inner core. In this case, the penetrating fluid from the outer core would reach the center long before any noticeable fingering occurs, and a different treatment, perhaps a high order perturbation expansion of the microphysical equations, would be appropriate.

The two fluid growth rate model (24) can be generalized to obtain appropriate boundary conditions for $n$ layers. In order for this approach to be valid, we make three assumptions: (1) every interface is unstable; (2) fluid layers are initially thick enough to allow the formation of bubbles and spikes; and (3) there is no significant distortion of the coherent structures due to shearing. Note that in any experimental setting, the likelihood that one of the assumptions (1) and (2) is violated increases rapidly with increasing $n$.

With these assumptions, and the convention that any external acceleration is applied downward or toward the center, the upward (+) and downward (−) edge trajectories are associated with bubble and spike tips, respectively. The drag coefficient appropriate for
(bubbles, spikes) is denoted $C^\pm$, which is a function of the local Atwood number, to be explained shortly.

The generalization of the two fluid drag/buoyancy model becomes straightforward if we recognize that a bubble or spike of fluid $k$ does not necessarily penetrate a pure phase region, but rather it moves through a mixture of the other $n - 1$ fluids. The density of this background fluid, which is needed for the calculation of the phenomenological coefficient $C^\pm$, is simply the sum of the densities of the individual fluids, weighted by the corresponding mixture fractions. Thus we define the mixture fraction, density, and velocity excluding fluid $k$,

$$
\beta'_k = \sum_{j \neq k} \beta_j = 1 - \beta_k,
\rho'_k = \frac{1}{\beta'_k} \sum_{j \neq k} \beta_j \rho_j,
\nu'_k = \frac{1}{\beta'_k} \sum_{j \neq k} \beta_j \nu_j,
$$

and a corresponding signed Atwood number for fluid $k$ relative to the background mixture,

$$
A_k = \frac{\rho_k - \rho'_k}{\rho_k + \rho'_k}.
$$

A bubble or spike does not necessarily move through an ambient medium, so we must express its velocity relative to the bulk velocity of the background mixture. The phenomenological model for the $k^\pm$ (bubble, spike) edge is then

$$
\frac{d}{dt} (V^\pm_k - \nu'_k) = A_k g(t) \pm \frac{C^\pm_k \rho'_k}{\rho_k + \rho'_k} \frac{|V^\pm_k - \nu'_k|(V^\pm_k - \nu'_k)}{L^\pm_k},
$$

where $L^\pm_k$ is a length scale representing the volume to frontal surface area ratio of the bubble or spike. Two definitions of this length scale have appeared in the literature. The most common method is to define $L^\pm_k$ to be the net displacement of the structure,

$$
L^\pm_k(t) = H^\pm_k(t) - H^\pm_k(0).
$$

When $n = 2$, this definition has the advantage of being independent of any two-phase flow quantities, thus allowing the drag/buoyancy model to be analyzed independent of any two-phase flow model. This advantage disappears when $n \geq 3$, as any structure will
in general experience a non-trivial variation in the mixture fraction, which will couple the drag/buoyancy model to the multiphase flow equations.

An issue which deserves further study is the coupling of the growth rate model to the mixing zone interior, a preliminary discussion of which appears in [12]. For this purpose, it may be preferable to relate \( L_k^\pm \) to the volume fraction gradient, which appears naturally in the multiphase flow equations,

\[
L_k^\pm(t) = \left( \frac{\partial \beta_k}{\partial h} \right)^{-1} \bigg|_{H_k^\pm(t),t}.
\]

5 Special Applications

5.1 Two Materials System

Applying the general \( n \) material closure to two-phase flow \((n = 2)\), for two materials, \( \sigma_{kj}^v \) was derived rigorously from the boundary conditions that at edge \( k \),

\[
\beta_k = 0, \quad v_k = V_k \text{ at } h = H_k(t)
\]

and has form \( \sigma_{kj}^v = \frac{V_k}{V_j} \), where \( V_j \) is the edge velocity of the \( j \)th material. For self similar flow, the ratio of the edge velocities of the \( k \)th material to the \( j \)th material has a specific expression [14]

\[
\sigma_{kj}^v = \frac{V_k}{V_j} = -\frac{1}{3}(\rho_j - \rho_k) + \frac{[\frac{1}{3}(\rho_j - \rho_k)^2 + 4\rho_k\rho_j]^{1/2}}{2\rho_j}.
\]

Substituting \( \sigma_{kj}^v \) into \( v_k^* \), we obtain

\[
v_1^* = v_2^* = \frac{\beta_2 V_2 v_1 + \beta_1 V_1 v_2}{\beta_1 V_1 + \beta_2 V_2}
\]

and

\[
p_1^* = p_2^* = \frac{\beta_2 \rho_2 p_1 + \beta_1 \rho_1 p_2}{\beta_1 \rho_1 + \beta_2 \rho_2}.
\]

In two-phase flow, there is only one independent variable \( \{v_k\} \) and one ordinary differential equation for it

\[
\frac{d \ln v_k}{d \beta_k} = \frac{\mu_{kj}^v}{\beta_j} - \frac{\mu_{jk}^v}{\beta_k} - \frac{1}{\beta_j}.
\]
Substituting the formula for \( \mu_{jk} \) in this expression, integrating over \( \beta_k \), and using the boundary conditions, we obtain the solution

\[
v_k = \frac{V_j|V_k|\beta_k}{|V_k|\beta_k + |V_j|\beta_j} = V_k\mu_{kj}^v.
\]  

(28)

These equations are the model equations obtained in the two-phase flow model. Therefore the general closure for \( n \) materials proposed in this paper has successfully reduced to the known two material closure, and the exact solution for this model can be found in [12].

### 5.2 Three Materials System

We now consider the case \( n = 3 \). For three materials the interfacial velocities have the form

\[
v_1^* = \frac{1}{1-\beta_1} \left[ \frac{\beta_2^2}{\beta_1\sigma_{12}^v + \beta_1} + \frac{\beta_3^2}{\beta_1\sigma_{13}^v + \beta_3} \right] v_1 + \frac{\sigma_{12}^v \beta_2}{\beta_1\sigma_{12}^v + \beta_1} v_2 + \frac{\sigma_{13}^v \beta_3}{\beta_1\sigma_{13}^v + \beta_3} v_3,
\]

(29)

\[
v_2^* = \frac{1}{1-\beta_2} \left[ \frac{\beta_1^2}{\beta_2\sigma_{21}^v + \beta_1} + \frac{\beta_3^2}{\beta_2\sigma_{23}^v + \beta_3} \right] v_1 + \frac{\beta_1}{\beta_2\sigma_{21}^v + \beta_2} v_2 + \frac{\beta_3}{\beta_2\sigma_{23}^v + \beta_3} v_3,
\]

(30)

\[
v_3^* = \frac{1}{1-\beta_3} \left[ \frac{\beta_1^2}{\beta_3\sigma_{31}^v + \beta_1} + \frac{\beta_2^2}{\beta_3\sigma_{32}^v + \beta_2} \right] v_1 + \frac{\beta_1}{\beta_3\sigma_{31}^v + \beta_3} v_2 + \frac{\beta_2}{\beta_3\sigma_{32}^v + \beta_3} v_3.
\]

(31)

The parameter \( \sigma_{kj}^v \) can be generalized from the form for two material systems and has the expression \( \sigma_{kj}^v \equiv V_k^+/V_j^- \). Substituting \( v^* \)'s into the differential equations gives the following equations

\[
\frac{\partial v_1}{\partial h} = -\frac{s}{h} v_1 + \frac{1}{1-\beta_1} \left[ \frac{\sigma_{12}^v \beta_2 (v_2 - v_1)}{\beta_1\sigma_{12}^v + \beta_2} + \frac{\sigma_{13}^v \beta_3 (v_3 - v_1)}{\beta_1\sigma_{13}^v + \beta_3} \right] \frac{\partial \beta_1}{\partial h},
\]

(32)

\[
\frac{\partial v_2}{\partial h} = -\frac{s}{h} v_2 + \frac{1}{1-\beta_2} \left[ \frac{\sigma_{21}^v \beta_1 (v_1 - v_2)}{\beta_2\sigma_{21}^v + \beta_1} + \frac{\sigma_{23}^v \beta_3 (v_3 - v_2)}{\beta_2\sigma_{23}^v + \beta_3} \right] \frac{\partial \beta_2}{\partial h},
\]

(33)

\[
\frac{\partial v_3}{\partial h} = -\frac{s}{h} v_3 + \frac{1}{1-\beta_3} \left[ \frac{\sigma_{31}^v \beta_1 (v_1 - v_3)}{\beta_3\sigma_{31}^v + \beta_1} + \frac{\sigma_{32}^v \beta_2 (v_2 - v_3)}{\beta_3\sigma_{32}^v + \beta_2} \right] \frac{\partial \beta_3}{\partial h}.
\]

(34)
in which only two of the $v_j$ are independent, while the other is given by the constraints

$$\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 = 0, \quad \beta_1 + \beta_2 + \beta_3 = 1.$$ 

The later constraint is equivalent to

$$(v_1^* - v_3^*) \frac{\partial \beta_1}{\partial h} + (v_2^* - v_3^*) \frac{\partial \beta_2}{\partial h} = 0.$$  \hspace{1cm} (35)$$

Clearly, these equations, coupled to the system (7), can be solved numerically. In particular, in a frame comoving frame with fluid 3, $\partial v'_3/\partial h = 0$. Let $v'_1 \equiv v_1 - v_3$ and $v'_2 \equiv v_2 - v_3$ represent the velocities of fluid 1 and fluid 2 relative to fluid 3. Then from Eqs. (19)-(21), we have

$$v'_1 = -\frac{\sigma^v_{12} \beta_2 (\beta_3 \sigma^v_{31} + \beta_1)}{\sigma^v_{31} \beta_1 (\beta_3 \sigma^v_{32} + \beta_2)} v'_2,$$  \hspace{1cm} (36)$$

$$\frac{\partial (v'_1 - v'_2)}{\partial h} = -\frac{s}{h} (v'_1 - v'_2) + \frac{1}{1 - \beta_1} \left[ \frac{\sigma^v_{12} \beta_2 (v'_2 - v'_1)}{\beta_1 \sigma^v_{12} + \beta_2} - \frac{\sigma^v_{13} \beta_3 v'_1}{\beta_1 \sigma^v_{13} + \beta_3} \right] \frac{\partial \beta_1}{\partial h}$$

$$- \frac{1}{1 - \beta_2} \left[ \frac{\sigma^v_{21} \beta_1 (v'_1 - v'_2)}{\beta_2 \sigma^v_{21} + \beta_1} - \frac{\sigma^v_{23} \beta_3 v'_2}{\beta_2 \sigma^v_{23} + \beta_3} \right] \frac{\partial \beta_2}{\partial h},$$ \hspace{1cm} (37)$$

together with Eq. (22). These equations, coupled to the system (7), can be solved numerically.

The effective interfacial pressure in three material system has the form

$$p^*_1 = \frac{1}{1 - \beta_1} \left[ \left( \frac{\beta_2^2 \rho_2}{\beta_1 \rho_1 + \beta_2 \rho_2} + \frac{\beta_3^2 \rho_3}{\beta_1 \rho_1 + \beta_3 \rho_3} \right) p_1 \right.$$  \hspace{1cm} (38)$$

$$+ \frac{\beta_2 \beta_1 \rho_1}{\beta_1 \rho_1 + \beta_2 \rho_2} p_2 + \frac{\beta_3 \beta_1 \rho_1}{\beta_1 \rho_1 + \beta_3 \rho_3} p_3 \right]$$

$$p^*_2 = \frac{1}{1 - \beta_2} \left[ \frac{\beta_1 \beta_2 \rho_2}{\beta_2 \rho_2 + \beta_1 V_1} p_1 + \left( \frac{\beta_2^2 \rho_2}{\beta_2 \rho_2 + \beta_1 \rho_1} + \frac{\beta_3^2 \rho_3}{\beta_2 \rho_2 + \beta_3 \rho_3} \right) p_2 \right.$$  \hspace{1cm} (39)$$

$$+ \frac{\beta_3 \beta_2 \rho_2}{\beta_2 \rho_2 + \beta_3 \rho_3} p_3 \right]$$

$$p^*_3 = \frac{1}{1 - \beta_3} \left[ \frac{\beta_1 \beta_3 \rho_3}{\beta_3 \rho_3 + \beta_1 \rho_1} p_1 + \frac{\beta_2 \beta_3 \rho_3}{\beta_3 \rho_3 + \beta_2 \rho_2} p_2 \right.$$  \hspace{1cm} (40)$$

$$+ \left( \frac{\beta_2^2 \rho_2}{\beta_3 \rho_3 + \beta_1 \rho_1} + \frac{\beta_3^2 \rho_3}{\beta_2 \rho_2 + \beta_3 \rho_3} \right) p_3 \right].$$

Again with these expressions, the three-layered fluid system can be solved uniquely.

As an example, we now consider a simple two fluid problem with an intermediate layer added: fluid 1, $0 < h < 0.41$, $\rho_1 = 1$; fluid 2, $0.41 < h < 0.51$, $\rho_2 = \sqrt{3}$; and
fluid 3, $0.51 < h < 1.0$, $\rho_3 = 3$. The acceleration $g$ is set to be 0.5. Calculations are carried out numerically. The dynamical evolution of the volume fraction of each fluid are plotted in Fig. 2. Fig. 2a shows the volume fraction of fluid 1, 2, and 3 at time $t = 3.4$ respectively; and Fig. 2b displays the volume fraction of each fluid at a later time $t = 4.9$, in which solid lines represent the volume fraction of fluid 1, dot-dashed lines are for fluid 2, and long-dashed lines corresponds to fluid 3. These results show that the effects of the intermediate layer on the growth rate of the mixed region agree well with the Youngs’ numerical simulation and experimental results [20]. This model can be applied to more complicated systems as well.

6 Conclusion

In this paper, we have proposed a model for the unstable mixing of $n$ parallel or concentric incompressible fluid layers. The approach to constructing this model is pairwise, based on a known two incompressible fluid mixing model. The problem complexity increases significantly in going from two to three fluids, but the increase in complexity is relatively...
small thereafter. We present a detailed study of the \( n = 3 \) problem, which displays all of the significant modeling issues applicable to arbitrary \( n \geq 3 \) while still being reasonably tractable. The results are in good agreement with Youngs’ numerical calculations and experimental results.

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**References**


Figure Captions

Fig. 1 An illustration of the mixing layers at the $k$th fluid interfaces.

Fig. 2 The dynamical evolution of the volume fraction of each fluid in three fluid system. Fig. 2a shows the volume fraction of each fluid at time $t = 3.4$; and Fig. 2b demonstrates the volume fraction of each fluid at time $t = 4.9$, in which solid lines represent the volume fraction of fluid 1, dot-dashed lines correspond to fluid 2, and long-dashed lines denote fluid 3.