Chapter 1  Introductory Models

Section 1. Mathematical Models

Mathematical Model

A mathematical model is a mathematical formulation of some class of real-world problems.

Example 1. Falling objects

In physics, the height $H$ of an object dropped off a building is modeled by the formula:

$$H = -16T^2 + H_0$$

$T$ time (seconds) elapsed

$H_0$ height of the building.
\[ H = -16T^2 + H_0 \]

\[ T = 1 \]

\[ H = -16(1)^2 + 100 = 84 \]

\[ T = 2 \]

\[ H = -16(2)^2 + 100 = 36 \]

\[ H = 0 \] solve for \( T \)

\[ 0 = -16T^2 + 100 \]

\[ \Rightarrow 16T^2 = 100 \Rightarrow T = \frac{100}{16} \]

\[ T = \sqrt{\frac{100}{16}} = \frac{10}{4} = 2.5 \]

Example 2. A problem of Relative Ages

Michael is three times the age of his sister. In 6 years, he will be only twice his sister's age.
M  age of Michael
S  age of his sister

M = 3S
M + 6 = 2(S + 6)

⇒ 3S + 6 = 2S + 12
⇒ S = 6
⇒ M = 3S = 18

Example 3. Another Problem of Relative Ages.

Alice is currently twice as old as her brother Bill. If twice the sum of their current ages is equal to the product of their ages 4 years ago. How old is Bill?

A  age of Alice
B  age of Bill
\[ A = 2B \]
\[ 2(A+B) = (A-4)(B-4) \]
\[ 2A + 2B = AB - 4A - 4B + 16 \]
\[ \Rightarrow AB - 6A - 6B + 16 = 0 \]
\[ \Rightarrow 2B^2 - 12B - 6B + 16 = 0 \]
\[ 2B^2 - 18B + 16 = 0 \]
\[ B^2 - 9B + 8 = 0 \]
\[ (B-1)(B-8) = 0 \]

\[ B = 1 \] or \[ B = 8 \]

Modeling Process

Real World

Verification

Conclusions, Predictions

Symbolize the problem

Interpretation

mathematical solution

mathematical theory and techniques

mathematical model
Section 1.2 Systems of Linear Equations

\[ H = -16T + H_0 \]

Example 1. Oil refinery Model.

- three oil refineries
  - first refinery: 1 barrel of petroleum
    - 20 gallons of heating oil
    - 10 gallons of diesel oil
    - 5 gallons of gasoline
  - second refinery
    - 4 gallons of heating oil
    - 14 gallons of diesel oil
    - 5 gallons of gasoline
  - third refinery
    - 4 gallons of heating oil
    - 5 gallons of diesel oil
    - 12 gallons of gasoline
<table>
<thead>
<tr>
<th></th>
<th>Refinery 1</th>
<th>Refinery 2</th>
<th>Refinery 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heating oil</td>
<td>20</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Diesel oil</td>
<td>10</td>
<td>14</td>
<td>5</td>
</tr>
<tr>
<td>Gasoline</td>
<td>5</td>
<td>5</td>
<td>12</td>
</tr>
</tbody>
</table>

Let $x_i$ be the number of barrels of petroleum used by each refinery. $x_i$.

Heating oil: $20x_1 + 4x_2 + 4x_3$

Diesel oil: $10x_1 + 14x_2 + 5x_3$

Gasoline: $5x_1 + 5x_2 + 12x_3$

Suppose the demand is 500 units of heating oil, 850 units of diesel oil, 1000 units of gasoline.

\[
\begin{align*}
20x_1 + 4x_2 + 4x_3 &= 500 \\
10x_1 + 14x_2 + 5x_3 &= 850 \\
5x_1 + 5x_2 + 12x_3 &= 1000
\end{align*}
\]

Tried-and-true.
Guess. \( x_1 = 25 \), \( x_2 = 25 \), \( x_3 = 25 \)

\[
20(25) + 4(25) + 4(25) = 700 \quad 500 \\
10(25) + 14(25) + 5(25) = 725 \quad 800 \\
5(25) + 5(25) + 12(25) = 550 \quad 1000
\]

Second try.

\( x_1 = 10 \), \( x_2 = 25 \), \( x_3 = 70 \)

<table>
<thead>
<tr>
<th></th>
<th>( x_1 )</th>
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<tbody>
<tr>
<td>heating oil</td>
<td>580</td>
<td>500</td>
<td></td>
</tr>
<tr>
<td>diesel</td>
<td>800</td>
<td>850</td>
<td></td>
</tr>
<tr>
<td>gasoline</td>
<td>1015</td>
<td>1000</td>
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</table>

Third try. \( x_1 = 5 \), \( x_2 = 35 \), \( x_3 = 70 \)

<table>
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<th>( x_1 )</th>
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</thead>
<tbody>
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<td></td>
</tr>
<tr>
<td>diesel oil</td>
<td>890</td>
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<tr>
<td>gasoline</td>
<td>1040</td>
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