

HW#3. Due 10/16

2.6 3a, 6c, 7, 9

3.1 2. 3. 23

3.2 a. 3b. 5 (for 3b).
6 (for 3b)

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10/4/12

Chp 3. Solving Systems of linear
equation

3.1 Determinants.

Recall:

quadratic equation
 $ax^2 + bx + c = 0$.

solution:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Consider the following system

$$ax + by = e \quad ")$$

$$cx + dy = f \quad ")$$

Multiply ~~(1)~~ " by d , \leftrightarrow by b .
then subtract

$$d(ax + by = e)$$

$$b(cx + dy = f).$$

$$\begin{array}{r} adx + bdy = de \\ - bcx + bdy = bf \\ \hline adx - bcx = (de - bf) \end{array}$$

$$x = \frac{de - bf}{ad - bc}$$

Substitute this in (1). $ad - bc$

$$y = \frac{af - ce}{ad - bc}$$

Ex: $\begin{cases} 2x - 3y = 4 \\ x + 2y = 9 \end{cases} \quad a=2, b=-3, e=4 \\ c=1, d=2, f=9.$

$$x = \frac{de - bf}{ad - bc} = 5.$$

$$y = \frac{af - ce}{ad - bc} = 2.$$

This denominator is called
determinant.

$$ad - bc.$$

Definition :

The determinant of the n -by- n matrix A is defined to be the denominator in the algebraic expression for the solution of the system $\underline{Ax = b}$.

A is the coefficient matrix

Ex.
$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \underline{b} = \begin{bmatrix} e \\ f \end{bmatrix}$$
$$\underline{Ax = b}.$$

$$\det(A) = ad - bc.$$

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

2×2

$$\det(\underline{A}) = a_{11}a_{22} - a_{12}a_{21}$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{array} \right.$$

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

$$\det(\underline{A}) = a_{11}a_{22} - a_{12}a_{21}$$

denominator

Numerator . x_1

$$a_{22}b_1 - a_{12}b_2 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$$

$$\underline{A}_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{23} \end{bmatrix}.$$

$\det(\underline{A}_1)$ numerator in x_1 .

$$a_{11}b_2 - a_{21}b_1 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

$$\underline{A}_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$$

$\det(\underline{A}_2)$ numerator in x_2 .

The solution becomes

$$x_1 = \frac{\det(\underline{A}_1)}{\det(\underline{A})}$$

$$x_2 = \frac{\det(\underline{A}_2)}{\det(\underline{A})}$$

Cramer's Rule $n \times n \underline{A}$

$$\text{Define } \underline{A}_i = \begin{bmatrix} a_1^c & a_2^c & \cdots & a_{i-1}^c & b & a_{i+1}^c & \cdots & a_n^c \end{bmatrix}$$

$$i=1, 2, \dots, n$$

matrix obtained from replacing the i th column of \underline{A} by the RHS vector \underline{b} .

The solution to $\begin{matrix} \underline{A} \\ \underline{x} \end{matrix} = \begin{matrix} \underline{b} \end{matrix}$ is

$$x_i = \frac{\det(A_{i,i})}{\det(A)}, \quad i=1, 2, \dots, n$$

Important:

as long as $\det(\underline{A})$ is nonzero,
cramer's rule provides a
unique solution to the system
 $\begin{matrix} \underline{A} \\ \underline{x} \end{matrix} = \begin{matrix} \underline{b} \end{matrix}$.

Ex: $\begin{cases} 2x - 3y = 4 \\ x + 2y = 9 \end{cases}$

$$\underline{A} = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$$
$$\underline{A}_1 = \begin{bmatrix} 4 & -3 \\ 9 & 2 \end{bmatrix}, \quad \underline{A}_2 = \begin{bmatrix} 2 & 4 \\ 1 & 9 \end{bmatrix}$$

$$\det(\underline{A}) = 2 \times 2 - (-3) \times 1 = 7$$

$$\det(\underline{A}_1) = \begin{vmatrix} 4 & -3 \\ 9 & 2 \end{vmatrix} = 4 \times 2 - (-3) \times 9 = 35$$

$$\det(\underline{A}_2) = \begin{vmatrix} 2 & 4 \\ 1 & 9 \end{vmatrix} = 2 \times 9 - 4 \times 1 = 14$$

Cramer's Rule.

$$x = \frac{\det(A_1)}{\det(A)} = \frac{35}{7} = 5$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{14}{7} = 2,$$

Theorem 1.

Let A be an $n \times n$ matrix and b be an arbitrary n -vector.

If $\det(A) \neq 0$, then the system of equations $Ax = b$ has a unique solution given by Cramer's Rule.

* If $\det(A) = 0$, but one or more $\det(A_{ij}) \neq 0$, then no solution is possible.

* If all $\det(A_{ij}) = 0$, and $\det(A) \neq 0$, solutions may be possible.

Proposition 1

If one row (column) of an $n \times n$ matrix \tilde{A} equals, or is a multiple of, another row (column), then $\det(\tilde{A}) = 0$.

$$\text{ex. } \tilde{A} = \begin{bmatrix} 2 & 4 \\ 12 & 24 \end{bmatrix} \quad \det(\tilde{A}) = 0.$$

3×3 matrix

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} 3 \times 3$$

$$a_{11} \quad a_{12} \quad a_{13} \quad a_{11} \quad a_{12} \\ a_{21} \quad a_{22} \quad a_{23} \quad a_{21} \quad a_{22} \\ a_{31} \quad a_{32} \quad a_{33} \quad a_{31} \quad a_{32} \quad 3 \times 5$$

$$\begin{aligned} \det(\tilde{A}) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} \\ &\quad + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\ &\quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{aligned}$$

Ex. 7a.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\det(A) = 1 \cdot 2 \cdot 1 + 1 \cdot 1 \cdot 3 + 1 \cdot 1 \cdot 1$$
$$- 1 \cdot 2 \cdot 3 - 1 \cdot 1 \cdot 1 - 1 \cdot 1 \cdot 1$$
$$= -2$$

triangular matrix.

square matrix.

A square matrix is upper triangular matrix if all entries below the main diagonal are all zeros.

$$\begin{bmatrix} 2 & 4 & 1 \\ 0 & 1 & 7 \\ 0 & 0 & 2 \end{bmatrix}$$

upper triangular matrix

A lower triangular matrix has 0's above the main diagonal

Proposition 2

If \tilde{A} is an upper or lower triangular matrix,

$$\det(\tilde{A}) = a_{11} a_{22} \cdots a_{nn},$$

the product of entries on main diagonal.

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 1 & 7 \\ 0 & 0 & 2 \end{bmatrix} \quad \det(A) = 2 \cdot 1 \cdot 2 \\ = 4.$$

Proposition 3.

The determinant of matrix product is the product of the determinants.

$$\det(\tilde{A}\tilde{B}) = \det(\tilde{A}) \det(\tilde{B})$$