

10/25/12

iteration

convergence condition

$$\cancel{\text{if}} \quad \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}| < |a_{ii}| \quad \text{if}$$

$$\text{or} \quad \sum_{\substack{i=1 \\ i \neq j}}^n |a_{ij}| < |a_{jj}|.$$

- ① change of variable
- ② Jacobi iteration.

row condition

→ Jacobi iteration

$$\|D_1\|_{\max} < 1$$

change of variable

$$\|D_2\| > 1$$

Condition number of a matrix

$$\frac{\|x\|}{\|x + \xi\|} \leq C(A) \frac{\|\xi\|}{\|A\|}$$

$$\underline{Ax = b}.$$

$$(A + \xi)(x + \xi) = b.$$

$$C(A) = \|A^{-1}\| \cdot \|A\|$$

condition number of A.

Ex.  $A = \begin{bmatrix} 20 & 4 & 4 \\ 10 & 14 & 5 \\ 5 & 5 & 12 \end{bmatrix}$   $C(A) = \|A\| \|A^{-1}\|$

$$A^{-1} = \begin{bmatrix} 0.05958 & -0.01166 & -0.015 \\ -0.03958 & 0.09167 & -0.025 \\ -0.00833 & -0.03333 & 0.1 \end{bmatrix}$$

(largest column sums.  
sum norm)

$$\|A\|_S = 35 \quad (\text{first column})$$

$$\|A^{-1}\|_S = 0.14 \quad (\text{third column})$$

$$C(\tilde{A}) = \|A\|_S \|A^{-1}\|_S = 4.9.$$

$$\frac{|x|}{|x+e|} \leq C(A) \frac{\|E\|}{\|A\|}$$

$$\begin{matrix} 5\% & 4.9 \times 5\% \\ \hookrightarrow & 24.5\% \end{matrix}$$

$$\text{if } C(A) = 20.$$

$$5\% \rightarrow 100\%$$

$$C(A) = 4000 \quad \text{ill condition}$$

$$\tilde{A} = \begin{bmatrix} 20 & 4 & 4 \\ 10 & 14 & \underline{5} \\ 5 & 5 & 12 \end{bmatrix}$$

$$\tilde{A} + \tilde{E} = \begin{bmatrix} 20 & 4 & 4 \\ 10 & 14 & 7 \\ 5 & 5 & 12 \end{bmatrix}$$

$$\tilde{E} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\|E\|_S}{\|A\|_S} = \frac{2}{35} \doteq 6\% \quad E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\|E\|_S}{\|X+E\|_S} \leq C(A) \frac{\|E\|_S}{\|A\|_S} = 4.9 \times \frac{2}{35} = 28\%$$

$$A' = A + E = \begin{bmatrix} 20 & 4 & 4 \\ 10 & 14 & 7 \\ 5 & 5 & 12 \end{bmatrix}.$$

$$A'x' = b \quad \text{altered system} \\ x' = \begin{bmatrix} 6.5 \\ 19.7 \\ 72.3 \end{bmatrix} \quad x = \begin{bmatrix} 4.8 \\ 33.8 \\ 67.2 \end{bmatrix}$$

$$e = x' - x = \begin{bmatrix} 1.7 \\ -13.5 \\ 4.8 \end{bmatrix}$$

$$\|e\|_S = 1.7 + 13.5 + 4.8 = 20$$

$$\|X+e\|_S = 6.5 + 19.7 + 72.3 = 98.5$$

$$\frac{\|e\|_S}{\|X+e\|_S} = 0.20\%$$

$$19a) \quad \tilde{A} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

condition number

$$\begin{bmatrix} \tilde{A} & \mathbb{I} \end{bmatrix}$$

$$\tilde{A}^{-1} = \frac{1}{\det(\tilde{A})} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \mathbb{I} & \tilde{A}^{-1} \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$$

$$\tilde{A} \tilde{A}^{-1} = \mathbb{I}$$

$$c(\tilde{A}) = \|\tilde{A}\|_S \|\tilde{A}^{-1}\|_S = 3 \times 7 = 21$$

5%  $\rightarrow$  105%  
input error

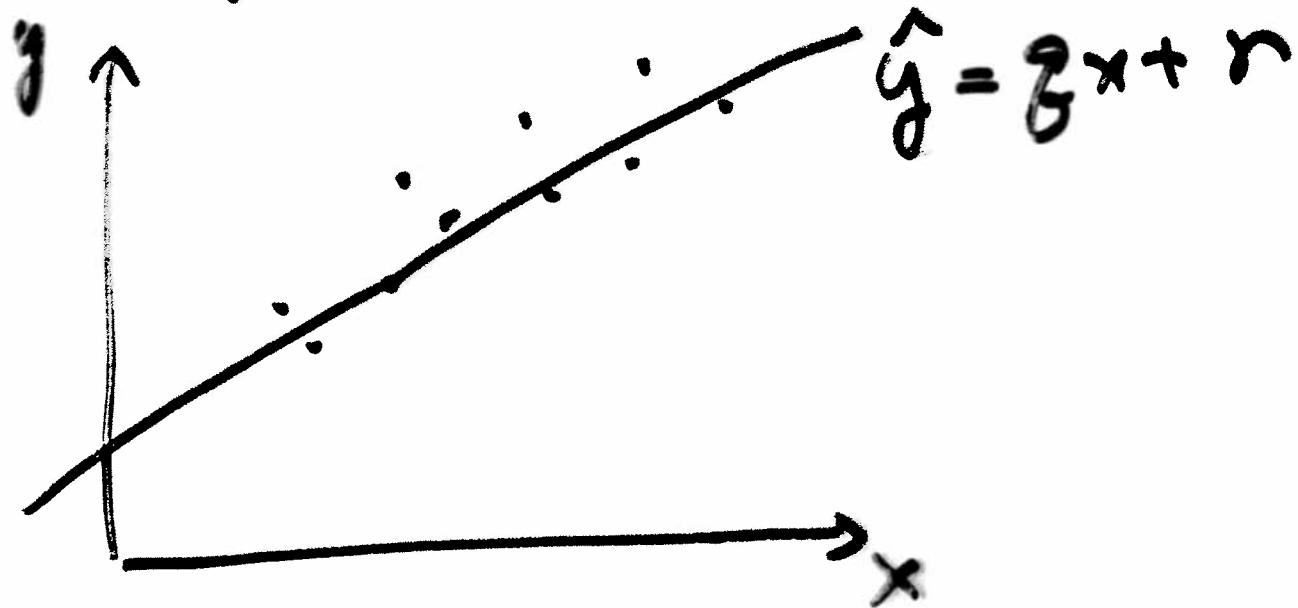
ill-conditioned.

## Chp 4. Sampling of Linear Models

### 4.2 linear regression

#### linear regression model

Given a set of points  $(x_1, y_1)$ ,  $(x_2, y_2)$ , ...,  $(x_n, y_n)$ . Find constant  $g$  and  $r$  such that the linear relation  $\hat{y}_i = g x_i + r$  gives the best possible fit for these points.



Non-linear regression

least square

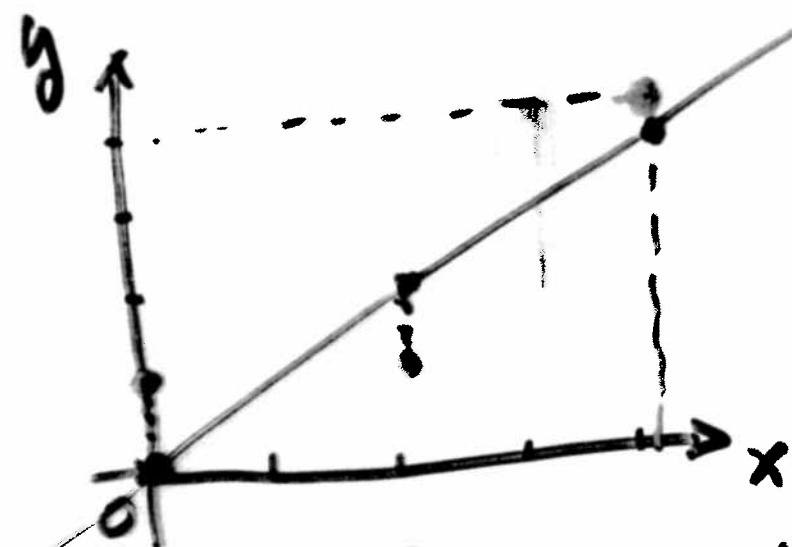
EX.

$x$	$y$
0	1
2	1
4	4.

$$\hat{y} = \frac{2}{3}x + r$$

$\downarrow$

$$r=0$$



$$\hat{y} = 8x$$

distance between two points

$(x_i, y_i)$   
d.

$$d = \sqrt{(x - x_i)^2 + (y - y_i)^2}$$

real point  $(x_i, y_i)$

predicted point  $(\hat{x}_i, \hat{y}_i)$ .

the square of errors.

$$(x_i - \hat{x}_i)^2 + (\hat{y}_i - y_i)^2$$

Sum of squares of errors.

$$SSE = \sum (\hat{y}_i - y_i)^2$$

smallest SSE.

- least square

minimum.

$$\frac{dSSE}{dg} = 0$$

g that minimizes  
SSE

$$\hat{Y} = gX$$

x	y	$\hat{y}$
0	1	0
2	1	$2g$
4	4	$4g$ .

$$SSE = \sum (\hat{y}_i - y_i)^2$$

$$= (0-1)^2 + (2g-1)^2 + (4g-4)^2$$

$$= 20g^2 - 36g + 18$$

g minimizes  
SSE

$$\frac{dSSE}{dg} = 40g - 36 = 0$$

$$\Rightarrow g = \frac{36}{40} = 0.9$$

$$SSE = 1.80$$

$$g = 0.9x$$

model

$$y = gx + r. \quad g, r.$$

SSE .

x	y	$\hat{y}$
0	1	$0g+r$
2	1	$2g+r$
4	4	$4g+r$ .

$$SSE = \sum (g_i - y_i)^2$$

$$= (r-1)^2 + (2g+r-1)^2 + (4g+r-4)^2$$

$$= 20g^2 + 3r^2 + 12gr - 36g - 12r + 18$$

$$\frac{\partial SSE}{\partial g} = 40g + 12r - 36 = 0$$

$$\frac{\partial SSE}{\partial r} = 6r + 12g - 12 = 0$$

$$\begin{cases} 40g + 12r = 36 \\ 12g + 6r = 12 \end{cases}$$

Cramer's Rule

$$g = \frac{|36 \quad 12|}{|40 \quad 12|} = 0.75$$

$$\quad \quad |12 \quad 6|$$

$$r = \frac{|40 \quad 36|}{|40 \quad 12|} = 0.5$$

$$\quad \quad |12 \quad 6|$$

$$\boxed{\hat{y} = 0.75x + 0.5}$$

$$\underline{SSE = 1.5}$$

general formula for regression  
model  $\hat{y} = g x$ .

$$\hat{y}_i = g x_i.$$

$$\begin{aligned} SSE &= \sum_i (\hat{y}_i - y_i)^2 \\ &= \sum_i (g x_i - y_i)^2 \\ &= \sum_i (g^2 x_i^2 - 2g x_i y_i + y_i^2) \end{aligned}$$

$$\frac{dSSE}{dg} = \sum (2x_i g - 2x_i y_i) = 0$$

$$\begin{aligned} \frac{dSSE}{dg} &= 2g \sum x_i^2 - 2 \sum (x_i y_i) = 0 \\ &= 2g \sum x_i^2 = 2 \sum (x_i y_i) \end{aligned}$$

$$\Rightarrow 2g \sum x_i^2 = 2 \sum (x_i y_i)$$

$$\Rightarrow g = \frac{\sum (x_i y_i)}{\sum x_i^2} = \frac{\bar{x} \cdot \bar{y}}{\bar{x} \cdot \bar{x}}$$

formula for regression model  
 $\hat{y} = g x$ .

Regression model  $\hat{y} = \beta x + r$ .

$$\hat{y}_i = \beta x_i + r.$$

$$SSE = \sum (\hat{y}_i - y_i)^2 \\ = \sum (\beta x_i + r - y_i)^2$$

$$\frac{\partial SSE}{\partial \beta} = 0. \quad \frac{\partial SSE}{\partial r} = 0$$

$$\beta = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{n \sum x_i^2 - (\sum x_i)^2}$$

$$r = \frac{(\sum y_i)(\sum x_i^2) - (\sum x_i)(\sum x_i y_i)}{n \sum x_i^2 - (\sum x_i)^2}$$

formula for regression model

$$\hat{y} = \beta x + r$$