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Chp 3. Solving systems of linear equations.

3.1 Determinants

2×2 matrix

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

3×3 matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Cramer's Rule

$$A \vec{x} = \vec{b} . \quad A \in \mathbb{R}^{n \times n}$$

$$A_i^c = [a_1^c \ a_2^c \ \dots \ a_{i-1}^c \ b \ a_{i+1}^c \ \dots \ a_n^c]$$

$$x_i = \frac{\det(A_i^c)}{\det(A)} . \quad i=1, 2, \dots, n$$

Triangular matrix
(lower, upper)

$$\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} & \dots \\ \vdots & \ddots & \ddots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ 0 & a_{22} & \dots \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_{nn} \end{bmatrix}$$

If \underline{A} is triangular matrix.

$$\det(\underline{A}) = a_{11}a_{22} \dots a_{nn}.$$

* $\det(\underline{A} \underline{B}) = \det(\underline{A}) \det(\underline{B})$

Finding eigenvalues

$$\det(\underline{A} - \lambda \underline{I}) = 0$$

characteristic polynomial.
roots of ✓

Finding eigenvectors for each eigenvalue

$$\text{Solve } (\underline{A} - \lambda \underline{I}) \underline{x} = \underline{0}$$

problem (1)

3.2 Elimination

Gaussian Elimination

1) transform the system into an upper triangular system

2) back substitution.

(may need to rearrange the order of equations)

$[A \ b]$ augmented matrix

LU decomposition $\tilde{A} = \tilde{L} \tilde{U}$

\tilde{U} : the final reduced sets of coefficients. (upper triangular matrix)

\tilde{L} multipliers used

$$\text{ex: } \begin{matrix} (2) - \frac{1}{2}(1) \\ (3) - 7(2) \end{matrix} \quad l_{21} = \frac{1}{2}, \quad l_{32} = 7$$

$$\det(\tilde{A}) = U_{11}U_{22} \cdots U_{nn}$$

the product of main diagonal entries

elimination by pivoting

(Gauss-Jordan Elimination)

$$\begin{bmatrix} A & b \\ I & x \end{bmatrix} \xrightarrow{\text{elimination}} \begin{bmatrix} I & x^* \\ I & x \end{bmatrix}$$

x^* is the solution to $Ax = b$.

3.3 Inverse of a Matrix

$$A^{-1} \quad AA^{-1} = I \quad A^{-1}A = I$$

A is invertible if A^{-1} exists.

$$Ax = b \quad x = A^{-1}b$$

For 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

if $\det(A) \neq 0$

general method.

$$\begin{bmatrix} A & I \\ I & A^{-1} \end{bmatrix} \xrightarrow{\substack{\text{elimination} \\ \text{by pivoting}}} \begin{bmatrix} I & A^{-1} \\ I & A^{-1} \end{bmatrix}$$

Properties of inverse

$$(\underline{A} \underline{B})^{-1} = \underline{B}^{-1} \underline{A}^{-1}$$

$$(\underline{A}^{-1})^{-1} = \underline{A}$$

$$(\underline{A}^T)^{-1} = (\underline{A}^{-1})^T$$

Fundamental theorem for solving $\underline{A} \underline{x} = \underline{b}$
equivalent statements

- ① $\underline{A} \underline{x} = \underline{b}$ has unique solution
- ② $\underline{A} \underline{x} = \underline{I} \underline{b}$ can be converted to
 $\underline{I} \underline{x} = \underline{A}^{-1} \underline{b}$ $[\underline{A} \ \underline{I}] \rightarrow [\underline{I} \ \underline{A}^{-1}]$

③ \underline{A} has an inverse

④ $\det(\underline{A}) \neq 0$.

If \underline{A} is $n \times n$ matrix with n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, and associated eigenvectors $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n$,

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n.$$

then $\underline{A} = \underline{U} \underline{\Lambda} \underline{U}^{-1}$. $\underline{A}^k = \underline{U} \underline{\Lambda}^k \underline{U}^{-1}$

$$\tilde{D}_{\lambda} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad \underline{x} = [x_1 \ x_2 \ \cdots \ x_n]$$

diagonal matrix.

$$\tilde{D}_{\lambda}^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}$$

34. Iteration.

heشتیت مدل $\Phi \underline{x} = \underline{R} \underline{x} + \underline{c}$.

Iteration:

$$\underline{x}^{(k+1)} = \underline{D} \underline{x}^{(k)} + \underline{c}$$

convergence requires $\|\underline{R}\| < 1$

Condition for convergence

$$\sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}| < |a_{ii}| \text{ or } \sum_{\substack{i=1 \\ i \neq j}}^n |a_{ij}| < |a_{jj}|$$

If condition satisfied, two ways to modify the system to use iteration method.

① Change of variable.

$$x'_1 = a_{11}x_1, x'_2 = a_{22}x_2, \dots$$

$$\tilde{x}' = D\tilde{x} + \tilde{c} \quad ||$$

② Jacobi iteration

$$x_1 = \frac{-a_{12}x_2 - a_{13}x_3 - \dots - a_{1n}x_n + b_1}{a_{11}}$$

$$x_2 = \frac{-a_{21}x_1 - a_{23}x_3 - \dots - a_{2n}x_n + b_2}{a_{22}}$$

⋮

$$\tilde{x} = D\tilde{x} + \tilde{c}$$

problem (3)

3.5 Condition number of matrix

$$C(A) = \|A^{-1}\| \|A\|$$

$$\frac{\|\underline{x}\|}{\|x + \underline{x}\|} \leq C(A) \frac{\|\underline{x}\|}{\|A\|}$$

small \rightarrow well-conditioned
 large \rightarrow ill-conditioned
 problem (?)

a). $\det(A - kI_n) = 0$

$$\begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0 \quad \lambda_1 = 4, \lambda_2 = 1$$

b). for $\lambda_1 = 4$.

$$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 - x_2 = 0$$

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \end{aligned}$$

$$U = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

for $\lambda_1 = 1$

$$\tilde{U}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

eigen vectors has to be non-zero

Vectors.

2). a). $\tilde{A}^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{6} \\ -\frac{5}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -1 & 0 & -\frac{1}{2} \end{bmatrix}$ $\tilde{A} \tilde{A}^{-1} = I$

b). $\tilde{x} = \tilde{A}^{-1} \tilde{b}$

c). Gauss Elimination.

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & -2 & 3 \\ 2 & 2 & -4 \end{bmatrix}$$

$$\begin{aligned} (2') &= (2) - (-1)(1) \\ (3') &= (3) - (-2)(1) \end{aligned}$$

$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & -3 & 4 \\ 0 & 0 & -2 \end{bmatrix} = \tilde{U}$$

$$\begin{aligned} l_{21} &= -1 \\ l_{31} &= -2 \end{aligned}$$

$$\tilde{L} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\tilde{L} \tilde{U} = \tilde{A}$$

$$\begin{aligned}
 2). \det(A) &= \det(\underline{\underline{L}}) \det(\underline{\underline{U}}) \\
 &= U_{11} U_{22} \cdots U_{33} \\
 &= (-1)(-3)(-2) = -6
 \end{aligned}$$

$$3). \underline{\underline{x}} = \underline{\underline{R}} \underline{\underline{x}} + \underline{\underline{b}}. \text{ (Jacobi)}$$

$$\underline{\underline{x}}' = \underline{\underline{D}} \underline{\underline{x}}' + \underline{\underline{b}}. \text{ (change of variable)}$$

a). Jacobi Iteration.

$$(1) \quad x_1 - \frac{4}{9}x_2 + \frac{1}{3}x_3 = -\frac{10}{9}$$

$$\frac{4}{8}x_1 + (2) - \frac{3}{8}x_3 = \frac{30}{8}$$

$$-\frac{3}{10}x_1 + \frac{3}{10}x_2 + (3) = \frac{20}{10}$$

↓

$$\boxed{
 \begin{aligned}
 x_1 &= \frac{4}{9}x_2 - \frac{1}{3}x_3 - \frac{10}{9} \\
 x_2 &= -\frac{1}{2}x_1 + \frac{3}{8}x_3 + \frac{30}{8} \\
 x_3 &= \frac{3}{10}x_1 - \frac{3}{10}x_2 + \frac{20}{10}
 \end{aligned}
 }$$

D.X.

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & \frac{4}{9} & -\frac{1}{3} \\ -\frac{1}{2} & 0 & \frac{1}{10} \\ \frac{5}{10} & -\frac{1}{10} & 0 \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} \frac{10}{9} \\ \frac{30}{9} \\ \frac{20}{10} \end{bmatrix}.$$

$$\underline{x} = D\underline{x} + \underline{b}$$

$$\|D\|_{\max} = \max \left\{ \frac{2}{9}, \frac{1}{8}, \frac{6}{10} \right\} = \frac{1}{8} < 1$$

b). $\boxed{\underline{x}^{(k+1)} = D\underline{x}^{(k)} + \underline{b}}$

$$\underline{x}^{(0)} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{x}^{(2)}$$

$$\underline{x}'' = D\underline{x}^{(0)} + \underline{b}.$$

$$\underline{x}^{(2)} = D\underline{x}'' + \underline{b}$$

$$4). C(\tilde{A}) = \|\tilde{A}^{-1}\| \|\tilde{A}\|$$

find \tilde{A}^{-1}

$$\tilde{A}^{-1} = \begin{bmatrix} 300 & -900 & 630 \\ -900 & 2880 & -2100 \\ 630 & -2100 & 1575 \end{bmatrix}$$

sum norm.

$$\|\tilde{A}\|_S = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}. \text{ (first)}$$

$$\|\tilde{A}^{-1}\|_S = 900 + 2880 + 2100 \quad (\text{second column}) \\ = 5880$$

$$C(\tilde{A}) = \|\tilde{A}^{-1}\|_S \|\tilde{A}\|_S \\ = 5880 \times \frac{47}{60} = 4606$$

\therefore ill-conditioned

$$\frac{|e|}{|x+2|} \leq C(\tilde{A}) \frac{\|E\|}{\|\tilde{A}\|}$$