

11/15/2012

Hw #5 Due 11/27/12

4.2: 2c, 5d

4.4: 1, 6, 17, 22

4.5: 1ac

5.1: 5ab, 8, 11ab, 21

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Regular M.C. $A =$ transition

$\tilde{A}P^* = P^*$ matrix

Solve $(\tilde{A} - I)P = 0$ for P^*

Absorbing M.C.

absorbing state $a_{ii} = 1$

#16

I, E, S, M

M = absorbing state

transition matrix

	Absorbing M	non-absorbing I E S		
M	1	0	0	0.25
I	0	0.5	0.25	0
E	0	0.5	0.5	0.25
S	0	0	0.25	0.25

4x4

↳

$$\tilde{A} = \begin{bmatrix} \tilde{I} & \tilde{R} \\ \tilde{0} & \tilde{Q} \end{bmatrix}$$

$$P^* = AP$$

$$\tilde{P}^k = \tilde{A}^k P$$

$$\tilde{A}^2 = \begin{bmatrix} \tilde{I} & \tilde{R} \\ \tilde{0} & \tilde{Q} \end{bmatrix} \begin{bmatrix} \tilde{I} & \tilde{R} \\ \tilde{0} & \tilde{Q} \end{bmatrix} = \begin{bmatrix} \tilde{I} + \tilde{R}\tilde{Q} & \tilde{I}\tilde{R} + \tilde{R}\tilde{Q} \\ \tilde{0} + \tilde{0}\tilde{Q} & \tilde{0}\tilde{R} + \tilde{Q}\tilde{Q} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{I} & \tilde{R} + \tilde{R}\tilde{Q} \\ \tilde{0} & \tilde{Q}^2 \end{bmatrix}$$

$$\tilde{A}^3 = \tilde{A}\tilde{A}^2 = \begin{bmatrix} \tilde{I} & \tilde{R} + \tilde{R}\tilde{Q} + \tilde{R}\tilde{Q}^2 \\ \tilde{0} & \tilde{Q}^3 \end{bmatrix}$$

$$\tilde{A}^k = \begin{bmatrix} \tilde{I} & \tilde{R} + \tilde{R}\tilde{Q} + \tilde{R}\tilde{Q}^2 + \dots + \tilde{R}\tilde{Q}^{k-1} \\ \tilde{0} & \tilde{Q}^k \end{bmatrix}$$

$$\tilde{R} + \tilde{R}\tilde{Q} + \tilde{R}\tilde{Q}^2 + \tilde{R}\tilde{Q}^3 + \dots + \tilde{R}\tilde{Q}^{k-1}$$

$$\Rightarrow \tilde{R}(I + Q + Q^2 + Q^3 + \dots + Q^{k-1})$$

$$= \tilde{R} \sum_{j=0}^{k-1} Q^j$$

$$\sum_{j=0}^{k-1} Q^j = (I - Q)^{-1}$$

fundamental matrix of absorbing M.C.

↳

Theorem 2

Let \tilde{N} be the fundamental matrix of an absorbing MC. when we start in nonabsorbing state j

- i) Entry (ij) of \tilde{N} is the expected number of times we visit the nonabsorbing state i (before absorption)
- ii) The j^{th} entry in the vector $\tilde{1}\tilde{N}$ gives the expected number of rounds before absorption
- iii) Entry (ij) in $\tilde{R}\tilde{N}$ is the probability of eventually ending up in the absorbing state i

Example

#16

	M	I	E	S
M	1	0	0	0.25
I	0	0.5	0.25	0
E	0	0.5	0.5	0.25
S	0	0	0.25	0.5

$$\tilde{R} = \begin{bmatrix} 0 & 0 & 0.25 \end{bmatrix}$$

$$\tilde{Q} = \begin{bmatrix} 0.5 & 0.25 & 0 \\ 0.5 & 0.5 & 0.25 \\ 0 & 0.25 & 0.5 \end{bmatrix}$$

fundamental matrix

$$\tilde{N} = (\tilde{I} - \tilde{Q})^{-1}$$

$$\tilde{I} - \tilde{Q} = \begin{bmatrix} 0.5 & -0.25 & 0 \\ -0.5 & 0.5 & -0.25 \\ 0 & -0.25 & 0.5 \end{bmatrix}$$

$$(\tilde{I} - \tilde{Q})^{-1} = \begin{matrix} & \begin{matrix} I & E & S \end{matrix} \\ \begin{matrix} I \\ E \\ S \end{matrix} & \begin{bmatrix} 6 & 4 & 2 \\ 8 & 8 & 4 \\ 4 & 4 & 4 \end{bmatrix} \end{matrix} = \tilde{N}$$

final inverse

$$\underline{1} \underline{N} = [1 \ 1 \ 1] \begin{bmatrix} 6 & 4 & 2 \\ 8 & 8 & 4 \\ 4 & 4 & 4 \end{bmatrix}$$

$$= [13 \ 16 \ 10]$$

Expected
 ↳ 13 round to go from I to M

$$\underline{RN} = [1 \ 1 \ 1]$$

Absorbly M.C.
 is done!!

4.5 Growth Model 

Ex. Age-Specific Population model

y = young, m = midlife, o = old
 0-2 yr, 2-4 yr, 4-6 yr

$$\underline{a} = \begin{bmatrix} y \\ m \\ o \end{bmatrix} \quad \underline{a}' = \begin{bmatrix} y' \\ m' \\ o' \end{bmatrix}$$

$y' = 4m + o$ — ie. 1 midlife make 4 y
 and 1 old make 1 y

$m' = 0.4y$ 40% of young will grow to midlife

$o' = 0.6m$ 60% midlife will grow old

↳ Leslie Model

$$\tilde{L} = \begin{bmatrix} \gamma & m & o \\ 0 & 4 & 1 \\ 0.4 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix} \begin{array}{l} \text{young} \\ \text{middle} \\ \text{old} \end{array}$$

$$\tilde{a}' = \tilde{L} \tilde{a}$$

more Age group will be

$$\begin{bmatrix} 0 & b_2 & b_3 & b_4 & \dots & b_n \\ p_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & p_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & p_3 & 0 & \dots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & p_{n-1} & 0 \end{bmatrix}$$

- b_i : number of offspring per individual in group i

- p_i : probability that an individual in group i survives one period to become a member of group $i+1$

The long-term population distribution $\tilde{L}^k \tilde{a}$ will be a multiple of the dominant eigenvector of \tilde{L} (associated with the largest eigenvalue)

The long-term growth rate

= dominant eigenvalue

70% young from \tilde{L} we get eigen-value

21% midlife

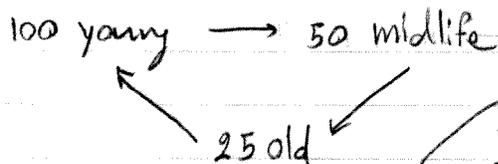
$$\lambda_1 = 1.334, \lambda_2 = -1.118, 0.152$$

9% old

$$u_1 = \begin{bmatrix} 0.697 \\ 0.209 \\ 0.094 \end{bmatrix}$$

Ex. A cyclic Leslie Model

$$\begin{array}{lcl} y' = & 40 & 0 \rightarrow 4y \\ m' = & 0.5y & 50\% y \rightarrow m \\ o' = & 0.5m & 50\% m \rightarrow o \end{array}$$



$$L = \begin{bmatrix} 0 & 0 & 4 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$$

$$\det(L - \lambda I) = 0$$

$$\begin{bmatrix} -\lambda & 0 & 4 \\ 0.5 & -\lambda & 0 \\ 0 & 0.5 & -\lambda \end{bmatrix} \begin{matrix} -\lambda & 0 \\ 0.5 & -\lambda \\ 0 & 0.5 \end{matrix}$$

$$= (-\lambda)^3 + 1 = 0$$

$$-\lambda^3 + 1 = 0$$

$$\lambda^3 - 1 = 0$$

$$(\lambda - 1)(\lambda^2 + \lambda + 1)$$

$$\lambda_1 = 1 \quad i = \sqrt{-1}$$

$$\lambda_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\lambda_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Theorem A Leslie model with matrix L will have a unique dominant eigenvalue that is a real number, and hence a stable long-term population distribution if some consecutive pair b_i, b_{i+1} of entries in the first row of L are both positive.