

11/20/12

Chp 5. Theory of systems of linear equations.

5.1 Null space and range of a matrix.

The Null space of matrix A is the set of vectors \underline{x} that are solutions to $A\underline{x} = \underline{0}$.

$$\text{Null}(A) = \{ \underline{x} \mid A\underline{x} = \underline{0} \}.$$

The Range of matrix A is the set of vectors \underline{b} such that $A\underline{x} = \underline{b}$ has a solution. $\text{Range}(A) = \{ \underline{b} \mid A\underline{x} = \underline{b} \text{ has a solution} \}$.

A vector space is any set V of vectors such that if $\underline{x}_1, \underline{x}_2$ are in V , then any linear combination $r\underline{x}_1 + s\underline{x}_2$ is also in V .

$$\text{Null}(A) = \{ \underline{x} \mid A\underline{x} = \underline{0} \}.$$

$\underline{x}_1, \underline{x}_2$ in $\text{Null}(A)$.

$$A\underline{x}_1 = \underline{0} \quad A\underline{x}_2 = \underline{0}$$

$r\underline{x}_1 + s\underline{x}_2$ in $\text{Null}(A)$?

$$A(r\underline{x}_1 + s\underline{x}_2) \stackrel{?}{=} \underline{0}$$

$$\underbrace{rA\underline{x}_1}_{\underline{0}} + \underbrace{sA\underline{x}_2}_{\underline{0}} = \underline{0} + \underline{0} = \underline{0}$$

$\Rightarrow \text{Null}(A)$ is a vector space.
so is $\text{range}(A)$.

chp 3. $A\underline{x} = \underline{b}$. Unique solution.

$$\det(A) \neq 0.$$

A is invertible.

this
in section .

$$A\underline{x} = \underline{b}$$

has an unique solution
(multiple)

or no solution

Theorem 1.

Let A be any m -by- n matrix.

(i) If $\text{Null}(A)$ contains one non zero vector x^0 , then $\text{Null}(A)$ contains an infinite number of vectors. rx^0

$$\begin{aligned} Ax^0 = 0 &\Rightarrow rAx^0 = 0 \\ &\rightarrow A(rx^0) = 0. \end{aligned}$$

rx^0 is in $\text{Null}(A)$.

(ii) If x^0 is in $\text{Null}(A)$ and x^* is a solution to $Ax = b$, then $x^* + x^0$ is also a solution to $Ax = b$.

$$\begin{aligned} Ax^0 = 0 \\ Ax^* = b \end{aligned} \quad \left. \vphantom{\begin{aligned} Ax^0 = 0 \\ Ax^* = b \end{aligned}} \right\} \text{add up.}$$

$$A(x^* + x^0) = 0 + b = b.$$

(iii) If x_1, x_2 are two different solutions to $Ax = b$, for some given b ,

then their difference $\underline{x}_1 - \underline{x}_2$ is a vector in $\text{Null}(A)$.

$$\begin{array}{l} A \underline{x}_1 = \underline{b} \\ A \underline{x}_2 = \underline{b} \end{array} \quad \rightarrow \quad A(\underline{x}_1 - \underline{x}_2) = \underline{b} - \underline{b} = \underline{0}$$

\Downarrow

$\underline{x}_1 - \underline{x}_2$ is in $\text{Null}(A)$

(iv) Given a solution \underline{x}^* to $A\underline{x} = \underline{b}$, then any other solution to this matrix equation can be written as

$$\underline{x}' = \underline{x}^* + \underline{x}^0 \text{ for some } \underline{x}^0 \text{ in } \text{Null}(A)$$

Given \underline{x}^* , \underline{x}' . let $\underline{x}^0 = \underline{x}' - \underline{x}^*$

by part (iii). \underline{x}^0 is in $\text{Null}(A)$.

(v) If $\text{Null}(A)$ consists of only the zero vector $\underline{0}$ then $A\underline{x} = \underline{b}$ has at most one solution, for any given \underline{b} .

proof. \underline{x}^* is the solution to $A\underline{x} = \underline{b}$.

by part (iv). any other solution \underline{x}' of

$A \underline{x} = \underline{b}$ can be expressed as $\underline{x}' = \underline{x}^* + \underline{x}^0$.

\underline{x}^0 is in $\text{Null}(A)$.

$\text{Null}(A)$ consists of just $\underline{0}$.

then $\underline{x}' = \underline{x}^* + \underline{0} = \underline{x}^*$.

$\Rightarrow \underline{x}^*$ is the only solution.

Theorem 2.

A m -by- n matrix A .

If $A \underline{x} = \underline{b}$ has two solutions, for some particular \underline{b} .

- (i). The Null space of A has an infinite number of vectors.
- (ii). For any \underline{b} , either $A \underline{x} = \underline{b}$ has no solution or an infinite number of solutions.

Ex 6a). $A = \begin{bmatrix} -3 & 4 & 1 \\ 2 & -1 & -1 \end{bmatrix}$

$A \tilde{x} = 0$. $\tilde{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{cases} -3x_1 + 4x_2 + x_3 = 0 \\ 2x_1 - x_2 + x_3 = 0 \end{cases}$$

$$\begin{bmatrix} -3 & 4 & 1 \\ 2 & -1 & 1 \end{bmatrix} \xrightarrow{G_1 = (2) + \frac{2}{3}(1)} \begin{bmatrix} -3 & 4 & 1 \\ 0 & \frac{5}{3} & \frac{5}{3} \end{bmatrix}$$

$$-3x_1 + 4x_2 + x_3 = 0 \Rightarrow -3x_1 + (-4x_3) + x_3 = 0$$

$$\frac{5}{3}x_2 + \frac{5}{3}x_3 = 0 \Rightarrow x_2 = -x_3$$

$$x_1 = -x_3$$

$$x_1 = -x_3$$

$$x_2 = -x_3$$

x_3 is free.

$$\tilde{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

\Rightarrow any scalar multiple of $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ is solution
I. $Ax = 0$

$$\text{Null}(A) = \left\{ r \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, r \in \mathbb{R} \right\}. \quad r \text{ is any real number}$$

$\text{Null}(A)$ is generated by the vector $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

Ex 7a) For 6a).

$$\underline{b} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.$$

$$A \underline{x} = \underline{b}$$

has a given solution

$$\underline{x}^* = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}.$$

$\underline{x}^* + \underline{x}^0$ is also a solution to $A \underline{x} = \underline{b}$. (Theorem 1).

$$\begin{aligned} \underline{x}' &= \underline{x}^* + r \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} + r \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}. \end{aligned}$$

the family of solution to $A \underline{x} = \underline{b}$.

$$x_3 = 5. \quad x_1, x_2.$$

$$10 + r = 5 \Rightarrow r = -5$$

EX. 6b). $A = \begin{bmatrix} 2 & 1 & 5 & 0 \\ 1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 1 \end{bmatrix}$

$Ax = 0$ Elimination by pivoting

$$\begin{bmatrix} 2 & 1 & 5 & 0 \\ 1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 1 \end{bmatrix} \xrightarrow{\substack{(1') = (1)/2 \\ (2') = (2) - (1) \\ (3') = (3) - (1)}} \begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{2} & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & -3 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

$(1'') = (1') - \frac{1}{2}(2'')$

$(2'') = (2') \cdot \frac{2}{3}$

$(3'') = (3') - \frac{1}{2}(2'')$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$Ax = 0$

$$\begin{cases} x_1 + 2x_3 + x_4 = 0 \\ x_2 + x_3 - 2x_4 = 0 \end{cases}$$

two equations
four unknown

x_3, x_4 free.

$\Rightarrow x_1 = -2x_3 - x_4$

$x_2 = -x_3 + 2x_4$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 - x_4 \\ -x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} -2x_3 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_4 \\ 2x_4 \\ 0 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{x} = r \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \text{ is in } \text{Null}(\underline{A}).$$

Two variable Null space.

$\text{Null}(\underline{A})$ is generated by these two vectors.

$$A \underline{x} = \underline{b}$$

Theorem 1.

$$\underline{x}^* + \underline{x}^0$$

solution to $A \underline{x} = \underline{b}$

EX 8. for (b).

$$\underline{x}^* = \begin{bmatrix} 10 \\ 10 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 30 \\ 30 \\ 20 \end{bmatrix}$$

$$\underline{x}' = \underline{x}^* + \underline{x}^0$$

$$= \begin{bmatrix} 10 \\ 10 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

is the family of all solutions
to $A \underline{x} = \underline{b}$.

Range of matrix.

EX 8.

$$20x_1 + 4x_2 = b_1$$

$$10x_1 + 14x_2 = b_2$$

$$5x_1 + 5x_2 = b_3$$

$$A \underline{x} = \underline{b}$$

Find a constraint on b_1, b_2, b_3 .

elimination by pivoting on $[A \ I]$

$$\left[\begin{array}{cc|cc} 20 & 4 & 1 & 0 & 0 \\ 10 & 14 & 0 & 1 & 0 \\ 5 & 5 & 0 & 0 & 1 \end{array} \right]$$

$$\downarrow$$
$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{120} & -\frac{1}{60} & 0 \\ 0 & 1 & -\frac{1}{24} & \frac{1}{12} & 0 \\ 0 & 0 & -\frac{1}{12} & -\frac{1}{3} & 1 \end{array} \right]$$

x_1

$$= \frac{2}{120} b_1 + \frac{1}{60} b_2$$

$$x_2 = -\frac{1}{24} b_1 + \frac{1}{12} b_2$$

$$0 = -\frac{1}{12} b_1 - \frac{1}{3} b_2 + b_3.$$

$$\Rightarrow b_3 = \frac{1}{12} b_1 + \frac{1}{3} b_2.$$

range constraint.

suppose $b_1 = 300$. $b_2 = 300$.

$$b_3 = \frac{1}{12} 300 + \frac{1}{3} 300 = 125$$

$$\left[\begin{array}{c} 300 \\ 300 \\ 125 \end{array} \right]$$