

12/4/12

Final Exam

12/17 2:15 - 5:00 pm

Simons 103 : A-LEE

Engineering 143 : LEN-Z

4.2 Linear regression



(x_i, y_i)

observation

\hat{y} model prediction.

SSE sum of squares of errors.

$$SSE = \sum (\hat{y}_i - y_i)^2$$

minimize SSE.

least-squares approach

Formula for $\hat{y} = g x$.

$$g = \frac{\sum x_i y_i}{\sum x_i^2}$$

Formula for $\hat{y} = g x + r$

$$g = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{n \sum x_i^2 - (\sum x_i)^2}$$

$$r = \frac{(\sum y_i)(\sum x_i^2) - (\sum x_i)(\sum x_i y_i)}{n \sum x_i^2 - (\sum x_i)^2}$$

shortcut for $\hat{y} = g x + r$

$$x'_i = x_i - \bar{x}$$

$$\bar{x} = \frac{1}{n} \sum x_i$$

$$\sum x'_i = 0$$

(average
of x_i)

$$g' = \frac{\sum x'_i y_i}{\sum x'^2_i} . \quad r' = \frac{\sum y_i}{n} .$$

$$g = g' , \quad r = r' - g \frac{\sum x_i}{n} .$$

4. Markov Chain

\tilde{A} transition matrix.

a_{ij} transition probability.
 $j \rightarrow i$

current next

	1	2	3	4	5	Current
next	1	2	3	4	5	
	1	2	3	4	5	columns,

column sum = 1

\tilde{P} current

\tilde{P}' next period of time.

$$\tilde{P}' = \tilde{A} \tilde{P}.$$

$$\tilde{P}^{(K)} = \tilde{A}^K \tilde{P}.$$

For some M.C. $\tilde{P}^{(K)}$ will converge
 to \tilde{P}^*
 (stable distribution)

$$\underline{P}^* = \underline{A} \underline{P}^*$$

regular M.C. has stable probability distribution \underline{P}^* .
all columns of \underline{A}^k will converge to \underline{P}^* .

How to find \underline{P}^* for regular M.C.

$$\underline{P}^* = \underline{A} \underline{P}^*$$

$$(\underline{A} - \underline{\lambda}) \underline{P}^* = \underline{0}$$

$$\text{solution to } (\underline{A} - \underline{\lambda}) \underline{x} = \underline{0}$$

constraint $\sum P_i = 1$

Absorbing M.C. (nonregular M.C.).
absorbing state $a_{ii} = 1$

How to analyse absorbing M.C.

① transition matrix.

List Ab states first. partition it

Ab NAb ← current

next

$$\begin{matrix} \text{Ab} & \left[\begin{array}{c|cc} I & ; & R \\ \hline \sim & \vdots & \sim \\ - & \vdots & - \\ \sim & \vdots & \sim \end{array} \right] \\ \text{NAb} & \left[\begin{array}{c|cc} D & ; & \sim \\ \hline \sim & \vdots & \sim \end{array} \right] \end{matrix}$$

② calculate the fundamental matrix

$$N = (I - Q)^{-1} \quad (= \sum_k Q^k)$$

③ Theorem 2. to answer questions.

$$\begin{matrix} N & . \\ \sim & \left[\begin{array}{c|c} & \\ \hline NAb & \end{array} \right] \end{matrix}$$

between
nonabsorbing
states.

$$\begin{matrix} \text{1} \\ \text{N} \end{matrix} \quad [E' \cdot \cdot \cdot] \left[\begin{matrix} \text{NAB} \\ \text{N} \end{matrix} \right]$$

$$= \left[\begin{matrix} \text{NAB} \\ \text{N} \end{matrix} \right]$$

nonabsorbing state to absorption

$$\begin{matrix} \text{R} \\ \text{N} \end{matrix} \quad \boxed{\text{probability}} \quad \begin{matrix} \text{NAB} \\ \text{NAB} \end{matrix}$$

$$\text{Ab} \left[\begin{matrix} \text{R} \\ \text{N} \end{matrix} \right] = \left[\begin{matrix} \text{NAB} \\ \text{NAB} \end{matrix} \right]$$

$\text{NAB} \rightarrow \text{Ab}$. probability

for example in class.

$$\begin{matrix} \text{R} \\ \text{N} \end{matrix} = [\cdot \cdot \cdot]$$

2 absorbing states.
 NAB

$$\begin{matrix} \text{R} \\ \text{N} \end{matrix} = \text{Ab} \left[\begin{matrix} \square \\ \square \end{matrix} \right]$$

4.5 Leslie Model.

age specific growth model

$$y, m, o.$$

$$y = 4m + o$$

$$m = 0.4y$$

$$o = 0.6m$$

$$\underline{L} = \begin{bmatrix} 0 & 4 & 1 \\ 0.4 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}$$

$$\underline{\alpha}' = \frac{1}{\lambda} \underline{\alpha}$$

generalise

$$\underline{L} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 & b_4 & \cdots & b_n \\ p_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & p_2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & p_3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & p_4 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & p_{n-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

b_i : number of offspring

p_i : surviving probability

long-term growth rate

= dominant eigenvalue of \tilde{L}

long-term population distribution

= ~~choose~~ a multiple of the dominant eigenvector.

$$\text{solve } \det(\tilde{L} - \lambda I) = 0$$

involving complex number

$$\begin{vmatrix} q_{11} & q_{12} & q_{13} & q_{11} & q_{12} \\ q_{21} & q_{22} & q_{23} & q_{21} & q_{22} \\ q_{31} & q_{32} & q_{33} & q_{31} & q_{32} \end{vmatrix}$$

$$= q_{11}q_{22}q_{33} + q_{12}q_{23}q_{31} + q_{13}q_{21}q_{32} - q_{13}q_{22}q_{31} - q_{11}q_{23}q_{32} - q_{12}q_{21}q_{33}$$

Some growth model is cyclic.

two consecutive age groups give birth to offspring. \Rightarrow unique long-term distribution

5.1 Null space and range of a matrix.

$$\text{Null}(A) = \{ \underline{x} \mid A\underline{x} = \underline{0} \}$$

$$\text{range}(A) = \{ \underline{b} \mid A\underline{x} = \underline{b} \text{ has solutions} \}$$

$$\text{col}(A) = \text{range}(A).$$

Vector space.

$$\underline{x}^0 \text{ in } \text{Null}(A)$$

$$A\underline{x}^0 = \underline{0}.$$

\underline{x}^* is a solution to $A\underline{x} = \underline{b}$.

then $\underline{x}^* + \underline{x}^0$ is also a solution to
 $A\underline{x} = \underline{b}$.

Determine range constraint
 $[A \ \underline{b}]$ elimination by pivoting

ex. $\begin{bmatrix} 20 & 4 & 1 & 0 & 0 \\ 10 & 14 & 0 & 1 & 0 \\ 3 & 5 & 0 & 0 & 1 \end{bmatrix} \quad Ax = b$

$\downarrow \quad \downarrow$

$$\left[\begin{array}{cc|ccc} 1 & 0 & \frac{7}{20} & -\frac{1}{60} & 0 \\ 0 & 1 & -\frac{1}{20} & \frac{1}{12} & 0 \\ 0 & 0 & -\frac{1}{12} & -\frac{1}{3} & 1 \end{array} \right] \quad \text{I}'x = M'b$$

$$\tilde{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{M}'$$

$$0 + 0 = -\frac{1}{12}b_1 + -\frac{1}{3}b_2 + b_3$$

\downarrow range constraint

5.2 basis for $\text{Null}(A)$ and range (A) :

basis: linear independent set of vectors that generate the vector space.

Method :
Elimination by pivoting on \tilde{A} .

$$\begin{matrix} A \\ \sim \end{matrix} \rightarrow \begin{matrix} A^* \\ \sim \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

~~pivoted columns : basis for range (\tilde{A})
 $\text{col}(\tilde{A})$~~

~~unpivoted columns : free variables.
write out solution to $\tilde{A}\tilde{x} = \underline{0}$
in terms of the free variables
↓ basis for $\text{Null}(\tilde{A})$.~~

dimension of a vector space V
= # of vectors in a basis for
 $\text{rank}(\tilde{A}) = \dim(\text{range}(\tilde{A}))$.

~~dim~~

$$\dim(\text{range}(Q)) + \dim(\text{Null}(Q)) = n$$

\Rightarrow # of columns