Simulate $\alpha$-Stable Random Variable and Estimate Stable Parameters Based on Market Data

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1. **Definition of $\alpha$-Stable Random Variable**
   - Definition
   - Characteristic Function

2. **Weron’s Algorithm for Generating Stable Random Variable**
   - Properties of Stable Law
   - Integral Form of Stable Law
   - Generation of Stable Random Variable

3. **Estimation of Stable Distribution Parameters**
   - The Method of Moments
   - Time Regression Method

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Simulate $\alpha$-Stable Random Variable
Definition

Given independent and identically distributed random variables $X_1, X_2, \ldots, X_n$ and $X$, then random variable $X$ is said to follow an $\alpha$-Stable distribution if there exists a positive constant $C_n$ and a real number $D_n$ such that the following relation holds:

$$X_1 + X_2 + \cdots + X_n = C_nX + D_n$$

where $=$ denotes equality in distribution and constant $C_n = n^{\frac{1}{\alpha}}$ determines the stability property.

- When $\alpha = 2$, it is the Gaussian case; when $0 < \alpha < 2$, we have the non-Gaussian case.
- There are three special cases with a closed form p.d.f.
  - the Gaussian case ($\alpha = 2$)
  - the Cauchy case ($\alpha = 1, \beta = 0$)
  - the Lévy case ($\alpha = \frac{1}{2}, \beta = \pm 1$)
\( \alpha \)-Stable distribution does not have closed form density function and is expressed by characteristic function (form A):

\[
\phi_{\text{stable}}(t; \alpha, \sigma, \beta, \mu) = E\left[e^{itX}\right]
\]

\[
= \begin{cases} 
\exp \left( i\mu t - |\sigma t|^\alpha \left(1 - i\beta \left(\text{sign}(t) \tan \frac{\pi \alpha}{2}\right)\right) \right) & \alpha \neq 1 \\
\exp \left( i\mu t - |\sigma t| \left(1 + i\beta^2 \frac{2}{\pi} \left(\text{sign}(t) \ln |t|\right)\right) \right) & \alpha = 1 
\end{cases}
\]

where

\[
\text{sign}(t) = \begin{cases} 
1, & t > 0 \\
0, & t = 0 \\
-1, & t < 0 
\end{cases}
\]

Four related parameters are:

- \( \alpha \): the index of stability or the shape parameter, \( \alpha \in (0, 2) \)
- \( \beta \): the skewness parameter, \( \beta \in [-1, 1] \)
- \( \sigma \): the scale parameter, \( \sigma \in (0, +\infty) \)
- \( \mu \): the location parameter, \( \mu \in (-\infty, +\infty) \)
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By transforming $\beta, \sigma$, there is an equivalent definition of $\alpha$-stable distribution and characteristic function.

**Definition**

A random variable $X$ is $\alpha$-stable with parameters $(\alpha, \beta, \sigma, \mu)$ if and only if its characteristic function (form B) is given by

$$
\ln \phi(t) = \begin{cases} 
  i\mu t - \sigma_2^\alpha |t|^{\alpha} \exp\left(-i\beta (\text{sign}t) \frac{\pi}{2} K(\alpha)\right) & \alpha \neq 1 \\
  i\mu t - \sigma_2 |t| \left(\frac{\pi}{2} + i\beta (\text{sign}t) \ln |t|\right) & \alpha = 1
\end{cases}
$$

where

$$
K(\alpha) = \alpha - 1 + \text{sign} (1 - \alpha) = \begin{cases} 
  \alpha & \alpha < 1 \\
  \alpha - 2 & \alpha > 1
\end{cases};
$$

and for $\alpha \neq 1$

$$
\tan \left(\beta \frac{\pi K(\alpha)}{2}\right) = \beta \tan \frac{\pi \alpha}{2}, \sigma_2 = \sigma \left(1 + \beta^2 \tan^2 \frac{\pi \alpha}{2}\right)^{\frac{1}{2\alpha}};
$$

and for $\alpha = 1$

$$
\beta_2 = \beta, \sigma_2 = \frac{2}{\pi} \sigma.
$$
Corollary

Any two admissible parameter quadruples \((\alpha, \beta, \sigma, \mu)\) and \((\alpha, \beta, \sigma', \mu')\) uniquely determine real numbers \(a > 0\) and \(b\) such that

\[
X (\alpha, \beta, \sigma, \mu) \overset{d}{=} aX (\alpha, \beta, \sigma', \mu') + b
\]

where

\[
a = \frac{\sigma}{\sigma'}, \quad b = \begin{cases} 
\mu - \mu' \frac{\sigma}{\sigma'} & \alpha \neq 1 \\
\mu - \mu' \frac{\sigma}{\sigma'} + \sigma \beta \frac{2}{\pi} \ln \frac{\sigma}{\sigma'} & \alpha = 1 
\end{cases}.
\]

- Consider the standard case \(X (\alpha, \beta, 1, 0)\) and transform it to the general case \(X (\alpha, \beta, \sigma, \mu) = aX (\alpha, \beta, 1, 0) + b\) with

\[
a = \sigma, \quad b = \begin{cases} 
\mu & \alpha \neq 1 \\
\mu + \sigma \beta \frac{2}{\pi} \ln \sigma & \alpha = 1 
\end{cases}.
\]

- There are three formulas regarding probability density function, cumulative density function and characteristic function

\[
f (-x, \alpha, \beta) = f (x, \alpha, -\beta)
\]
\[
F (-x, \alpha, \beta) = 1 - F (x, \alpha, -\beta)
\]
\[
\phi (-t, \alpha, \beta) = \phi (t, \alpha, -\beta)
\]
Assume \( f(x, \alpha, \beta) \) and \( \phi(t, \alpha, \beta) \) are the density function and characteristic function (form B) of stable random variable \( X(\alpha, \beta, 1, 0) \), then according to the inversion formula of characteristic function

\[
f(x, \alpha, \beta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t, \alpha, \beta) \, dt
\]

\[
= \frac{1}{2\pi} \left( \int_{0}^{\infty} e^{-itx} \phi(t, \alpha, \beta) \, dt + \int_{0}^{\infty} e^{itx} \phi(-t, \alpha, \beta) \, dt \right).
\]

Since \( e^{-itx} \phi(t, \alpha, \beta) = e^{itx} \phi(-t, \alpha, \beta) \), then

\[
f(x, \alpha, \beta) = \frac{1}{\pi} \Re \int_{0}^{\infty} e^{-itx} \phi(t, \alpha, \beta) \, dt
\]

\[
= \frac{1}{\pi} \Re \int_{0}^{\infty} e^{itx} \phi(-t, \alpha, \beta) \, dt = \frac{1}{\pi} \Re \int_{0}^{\infty} e^{itx} \phi(t, \alpha, -\beta) \, dt.
\]

Finally, the integral form of the density function is

\[
f(x, \alpha, \beta) = \frac{1}{\pi} \Re \int_{0}^{\infty} \exp \left( -itx - t^{\alpha} \exp \left( -i \frac{\pi}{2} \beta K(\alpha) \right) \right) \, dt
\]

in the case \( \alpha \neq 1 \), and

\[
f(x, 1, \beta) = \frac{1}{\pi} \Re \int_{0}^{\infty} \exp \left( -itx - \frac{\pi}{2} t - i\beta t \ln t \right) \, dt
\]

in the case \( \alpha = 1 \).
Theorem

Let

\[ \epsilon(\alpha) = \text{sign}(1 - \alpha), \quad \gamma_0 = -\frac{\pi}{2} \beta_2 \frac{K(\alpha)}{\alpha}, \]

\[ C(\alpha, \beta_2) = 1 - \frac{1}{4} \left(1 + \beta \frac{K(\alpha)}{\alpha}\right) (1 + \epsilon(\alpha)), \]

\[ U_\alpha(\gamma, \gamma_0) = \left(\frac{\sin \alpha (\gamma - \gamma_0)}{\cos \gamma}\right)^{\frac{\alpha}{1-\alpha}} \frac{\cos (\gamma - \alpha (\gamma - \gamma_0))}{\cos \gamma}, \]

\[ U_1(\gamma, \beta_2) = \frac{\frac{\pi}{2} + \beta_2 \gamma}{\cos \gamma} \exp \left(\frac{1}{\beta_2} \left(\frac{\pi}{2} + \beta_2 \gamma\right) \tan \gamma\right), \]

then the cumulative function of a standard stable distribution can be written as:

\[ F(x, \alpha, \beta_2) = C(\alpha, \beta_2) + \frac{\epsilon(\alpha)}{\pi} \int_{\gamma_0}^{\frac{\pi}{2}} \exp \left(-x^{1-\alpha} U_\alpha(\gamma, \gamma_0)\right) d\gamma, \quad \alpha \neq 1, \quad x > 0 \]

\[ F(x, 1, \beta_2) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp \left(-\exp \left(-\frac{x}{\beta_2}\right) U_1(\gamma, \beta_2)\right) d\gamma, \quad \alpha = 1, \quad \beta_2 > 0. \]
Theorem

Let $\gamma_0$ and $U_\alpha(\gamma, \gamma_0)$ be defined as above, for $\alpha \neq 1$, $\gamma_0 < \gamma < \frac{\pi}{2}$, $X$ is a $S_\alpha(1, \beta_2, 0)$ random variable iff for $x > 0$

$$\frac{1}{\pi} \int_{\gamma_0}^{\frac{\pi}{2}} \exp \left( -x \frac{\alpha-1}{\alpha} U_\alpha(\gamma, \gamma_0) \right) d\gamma = \begin{cases} P(0 < X \leq x) & \alpha < 1 \\ P(X \geq x) & \alpha > 1 \end{cases}$$

Proof:

- $0 < \alpha < 1$

$$F(x, \alpha, \beta_2) = P(X \leq x) = \frac{1 - \beta_2}{2} + \frac{1}{\pi} \int_{\gamma_0}^{\frac{\pi}{2}} \exp \left( -x \frac{\alpha-1}{\alpha} U_\alpha(\gamma, \gamma_0) \right) d\gamma$$

$$= \frac{1 - \beta_2}{2} + P(0 < X \leq x)$$

given that for $\alpha < 1$, $\frac{1 - \beta_2}{2} = P(X \leq 0)$.

- $1 < \alpha < 2$

$$F(x, \alpha, \beta_2) = P(X \leq x) = 1 - \frac{1}{\pi} \int_{\gamma_0}^{\frac{\pi}{2}} \exp \left( -x \frac{\alpha-1}{\alpha} U_\alpha(\gamma, \gamma_0) \right) d\gamma$$

$$= 1 - P(X \geq x)$$

This completes the proof.
Theorem

Let $\gamma_0$ be defined as above, $\gamma$ be uniformly distributed on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $W$ be an independent exponential random variable with mean 1. Then

$$X = \frac{\sin \alpha (\gamma - \gamma_0)}{(\cos \gamma)^{\frac{1}{\alpha}}} \left(\frac{\cos (\gamma - \alpha (\gamma - \gamma_0))}{W}\right)^{\frac{1-\alpha}{\alpha}}$$  

(1)

is $S_\alpha (1, \beta_2, 0)$ for $\alpha \neq 1$.

$$X = \left(\frac{\pi}{2} + \beta_2 \gamma\right) \tan \gamma - \beta_2 \log \left(\frac{W \cos \gamma}{\frac{\pi}{2} + \beta_2 \gamma}\right)$$  

(2)

is $S_1 (1, \beta_2, 0)$. 

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Proof.

Assume \( a(\gamma) = \left( \frac{\sin \alpha(\gamma-\gamma_0)}{\cos \gamma} \right)^{\frac{1}{1-\alpha}} \frac{\cos(\gamma-\alpha(\gamma-\gamma_0))}{\cos \gamma} \), then (1) is \( \left( \frac{\alpha(\gamma)}{W} \right)^{\frac{1-\alpha}{\alpha}} \).

When \( 0 < \alpha < 1 \), (1) implies that \( X > 0 \) iff \( \gamma > \gamma_0 \). Since \( \frac{1-\alpha}{\alpha} > 0 \)

\[
P(0 < X \leq x) = P(0 < X \leq x, \gamma > \gamma_0)
\]

\[
= P \left( 0 < \left( \frac{\alpha(\gamma)}{W} \right)^{\frac{1-\alpha}{\alpha}} \leq x, \gamma > \gamma_0 \right) = P \left( W \geq x^{\frac{\alpha}{\alpha-1}} a(\gamma), \gamma > \gamma_0 \right)
\]

\[
= P \left( W \geq x^{\frac{\alpha}{\alpha-1}} a(\gamma) \right) P(\gamma > \gamma_0) = E_\gamma \exp \left( -x^{\frac{\alpha}{\alpha-1}} a(\gamma) \right) 1\{\gamma > \gamma_0\}
\]

\[
= \int_{\gamma_0}^{\frac{\pi}{2}} \exp \left( -x^{\frac{\alpha}{\alpha-1}} a(\gamma) \right) \frac{1}{\pi} d\gamma
\]

Given \( a(\gamma) = U_\alpha(\gamma, \gamma_0) \), we have proved \( X \sim S_\alpha(1, \beta_2, 0) \).

And with the similar method, we can prove the case \( 1 < \alpha < 2 \) and the case \( \alpha = 1 \).
Algorithm:

- Generate a random variable $U$ uniformly distributed on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and an independent exponential random variable $E$ with mean 1.

- For $\alpha \neq 1$, compute
  \[
  X = S_{\alpha, \beta} \frac{\sin (\alpha (U + B_{\alpha, \beta}))}{(\cos (U))^{\frac{1}{\alpha}}} \left( \frac{\cos (U - \alpha (U + B_{\alpha, \beta}))}{E} \right)^{\frac{1-\alpha}{\alpha}},
  \]
  where $B_{\alpha, \beta} = \frac{\arctan (\beta \tan \frac{\pi \alpha}{2})}{\alpha}$, and $S_{\alpha, \beta} = (1 + \beta^2 \tan^2 \frac{\pi \alpha}{2})^{\frac{1}{2\alpha}}$.

- For $\alpha = 1$, compute
  \[
  X = \frac{2}{\pi} \left[ (\frac{\pi}{2} + \beta U) \tan U - \beta \log \left( \frac{\pi}{2} E \cos U \right) \right].
  \]

- Generalize scale and location $Y = \begin{cases} 
  \sigma X + \mu, & \alpha \neq 1 \\
  \sigma X + \frac{2}{\pi} \beta \sigma \log \sigma + \mu, & \alpha = 1
  \end{cases}$.
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Simulate $\alpha$-Stable Random Variable
Given sample of observed data $X = \{x_1, x_2, \cdots, x_N\}$ and assume it is directed by $\alpha$-stable distribution. Then, define the \textit{sample characteristic function}

$$\hat{\phi}_X (u) = \frac{1}{N} \sum_{j=1}^{N} e^{iux_j}.$$ 

By the law of large number, $\hat{\phi}_X (u)$ is a consistent estimator of the characteristic function $\phi_X (u)$.

By simple transformation, we have for all $\alpha$

$$|\phi_X (u)| = \exp (-\sigma^{\alpha} |u|^\alpha).$$

Thus

$$-\log |\phi_X (u)| = \sigma^{\alpha} |u|^\alpha.$$ 

Assume $\alpha \neq 1$, choose two different nonzero values $u_k$, $k = 1, 2$

$$-\log |\hat{\phi}_X (u_k)| = \sigma^{\alpha} |u_k|^\alpha.$$
Solve these two equations and get $\hat{\alpha}$, $\hat{\sigma}$

$$\hat{\alpha} = \frac{\log \left( \frac{\log |\hat{\phi}(u_1)|}{\log |\hat{\phi}(u_2)|} \right)}{\log \left| \frac{u_1}{u_2} \right|}$$

$$\log \hat{\sigma} = \frac{\log |u_1| \log \left( - \log |\hat{\phi}(u_2)| \right) - \log |u_2| \log \left( - \log |\hat{\phi}(u_1)| \right)}{\log \left| \frac{u_1}{u_2} \right|}.$$

The estimation of $\hat{\beta}$ and $\hat{\mu}$ based on the imaginary and real parts of the characteristic function

$$Re (\phi_X (u)) = \exp (- |\sigma \mu|^\alpha) \cos \left( \mu u + |\sigma u|^\alpha \beta (\text{sign} u) \tan \frac{\pi \alpha}{2} \right),$$

$$Im (\phi_X (u)) = \exp (- |\sigma \mu|^\alpha) \sin \left( \mu u + |\sigma u|^\alpha \beta (\text{sign} u) \tan \frac{\pi \alpha}{2} \right).$$

Then, we have

$$\left( \arctan \frac{Im (\phi_X (u))}{Re (\phi_X (u))} \right) = \mu u + |\sigma u|^\alpha \beta (\text{sign} u) \tan \frac{\pi \alpha}{2}.$$
Based on $\hat{\alpha}$, $\hat{\sigma}$ and two different nonzero values $u_k$, $k = 3, 4$, we can solve the system of equations to obtain estimation of $\hat{\beta}$ and $\hat{\mu}$.

\[
\hat{\mu} = \frac{u_4^\hat{\alpha} \arctan \frac{\text{Im}(\phi_X(u_3))}{\text{Re}(\phi_X(u_3))} - u_3^\hat{\alpha} \arctan \frac{\text{Im}(\phi_X(u_4))}{\text{Re}(\phi_X(u_4))}}{u_3 u_4^\hat{\alpha} - u_4 u_3^\hat{\alpha}}
\]

\[
\hat{\beta} = \frac{u_4 \arctan \frac{\text{Im}(\phi_X(u_3))}{\text{Re}(\phi_X(u_3))} - u_3 \arctan \frac{\text{Im}(\phi_X(u_4))}{\text{Re}(\phi_X(u_4))}}{\hat{\sigma}^\hat{\alpha} \tan \frac{\pi \hat{\alpha}}{2} (u_4 u_3^\hat{\alpha} - u_3 u_4^\hat{\alpha})}.
\]

In this estimation, the values are $u_1 = 0.2$, $u_2 = 0.8$, $u_3 = 0.1$ and $u_4 = 0.4$ are proposed in the simulation study.
Based on the equations

\[
\log \left( - \log |\phi_X(u)|^2 \right) = \log 2\sigma^\alpha + \alpha \log |u|
\]

\[
\left( \arctan \frac{\text{Im}(\phi_X(u))}{\text{Re}(\phi_X(u))} \right) = \mu u + \sigma^\alpha |u|^\alpha \beta \text{sign}(u) \tan \frac{\pi \alpha}{2}.
\]

By regression \( y_k = \log \left( - \log |\phi_X(u_k)|^2 \right) \) and \( w_k = \log |u_k| \) in the model

\[
y_k = aw_k + b + \epsilon_k
\]

propose \( u_k = \frac{\pi k}{25}, \ k = 1, 2, \ldots, K, \) then

\[
\hat{\alpha} = a, \quad \hat{\sigma} = \left( \frac{1}{2} e^b \right)^{\frac{1}{a}}.
\]
By regression $y_l = \left(\arctan \frac{\text{Im}(\phi_X(u_l))}{\text{Re}(\phi_X(u_l))}\right) \frac{1}{u_l}$ and $w_l = \frac{|u_l|^\alpha}{u_l} \text{sign} u_l$ in the model

$$y_l = aw_l + b + \epsilon_l$$

propose $u_l = \frac{\pi l}{50}, \ l = 1, 2, \cdots, L$, then

$$\hat{\mu} = b, \ \hat{\beta} = \frac{a}{\hat{\sigma} \hat{\alpha} \tan \frac{\pi \hat{\alpha}}{2}}.$$ 

A very important step is normalization with initial estimation

$$\sigma_0 = \frac{x_{0.72} - x_{0.28}}{1.654}, \ \mu_0 = E [X_{0.25-0.75}]$$

where $x_f$ is the $f$ sample quantile. Then the normalized data is

$$x_j' = \frac{x_j - \mu_0}{\sigma_0}.$$
Assume the estimated four parameters from regression of $x_j$ are

$$\alpha', \beta', \sigma', \mu'. \]

Then the stable parameters of original data are

$$\alpha = \alpha'$$

$$\beta = \beta'$$

$$\sigma = \sigma' \sigma_0$$

$$\mu = \begin{cases} 
\mu' \sigma_0 + \mu_0 & \alpha \neq 1 \\
\mu' \sigma_0 + \mu_0 - \sigma \beta \frac{2}{\pi} \ln \sigma_0 & \alpha = 1 
\end{cases}$$
References

- Rafel Weron, “Correction to: On the Chambers-Mallows-Stuck method for simulating skewed stable random variables”

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