Monotone convergence of finite element approximations of obstacle problems

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Abstract

The purpose of the work is to study the monotone convergence of numerical solutions of obstacle problems under mesh refinement when the obstacle is convex. We prove monotone convergence of piecewise linear finite element approximations for one-dimensional obstacle problems. We demonstrate by giving an example that such monotone convergence will not hold in the two-dimensional case.

Keywords: Obstacle problem; Monotone convergence; Finite element approximation

1. Introduction

The obstacle problem that we consider here can be described as follows: find the equilibrium position \( u = u(x) \), \( x \in \Omega \subset \mathbb{R}^2(\mathbb{R}^1) \) of an elastic membrane (string) constrained to lie above a given obstacle \( \Psi = \Psi(x) \). It is solved by the unique solution of the minimization problem

\[
\min_{v \in K} \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx,
\]

where \( K \) is a convex set of functions in \( H^1_0(\Omega) \) greater than or equal to \( \Psi \), i.e., \( K = \{v \in H^1_0(\Omega) : v \geq \Psi \text{ in } \Omega \} \). It is well known that this problem is equivalent to a variational inequality one, of finding \( u \in K \) such that

\[
a(u, v - u) \geq 0 \quad \text{for all } v \in K,
\]

where \((\cdot, \cdot)\) is the \( L^2 \)-inner product and taking \( a(\cdot, \cdot) \) to be the Dirichlet form:

\[
a(v, w) = \int_{\Omega} \nabla v \cdot \nabla w \, dx, \quad v, w \in H^1_0(\Omega).
\]

Obstacle problems are a type of free boundary problem. They are of interest both for their intrinsic beauty and for the wide range of applications they have in subjects from physics to finance. Many important problems can be formulated...
by transformation to an obstacle problem, e.g., the filtration dam problem [7], the Stefan problem [7], the subsonic flow problem [4], the American options pricing model [8]. The basic properties of the solution, including existence and uniqueness, were established by Lions and Stampacchia [5].

Given a grid \( G_h \), let \( \tilde{V}_h = \tilde{V}_h(G_h) \) denote the space of continuous piecewise linear functions over \( G_h \). Take \( V_h = V_h \cap H^1_0(\Omega) \). Let \( \psi_h \in \tilde{V}_h \) a discrete approximation of the continuous obstacle \( \psi \), and take \( K_h = \{ v_h \in V_h : v_h \geq \psi_h \} \). The discrete approximation of \( u \) is given by \( u_h \in K_h \), such that

\[
a(u_h, v_h - u_h) \geq 0 \quad \text{for all } v_h \in K_h. \tag{2}
\]

If we take \( \psi_h = \pi_h(\psi) \), where \( \pi_h \) is the piecewise linear interpolation operator, \( \psi_h \) monotonically increases as \( h \to 0 \) due to the convexity of \( \psi \). Now a natural question is whether the corresponding solution \( u_h \) increases monotonically as \( h \to 0 \), which seems the case intuitively. However, to our surprise the monotone convergence holds in one-dimensional space, but not in the two-dimensional case. The main tool for proving the monotone convergence in the one-dimensional case is the discrete monotonicity principle [3].

2. Monotone convergence in 1D solutions

The following lemma is elementary.

**Lemma 1.** Let \( u \) be a linear function on \([a, b]\). We define a continuous piecewise linear function \( v \) on \([a, b]\) such that \( v \) is linear on \([a, c]\) and \([c, b]\), and \( v(a) = u(a), v(b) = u(b) \), where \( a < c < b \). Then

\[
\int_a^b \left( \frac{dv}{dx} \right)^2 \, dx \geq \int_a^b \left( \frac{du}{dx} \right)^2 \, dx.
\]

Before we prove the monotone convergence theorem, we state the discrete monotonicity principle [9,10,3] as follows.

**Theorem 2** (Discrete Monotonicity Principle). Let \( \psi^1_h, \psi^2_h \in \tilde{V}_h \) be two discrete obstacles. Define \( K_1 = \{ v_h \in V_h : v_h \geq \psi^1_h \} \) and \( K_2 = \{ v_h \in V_h : v_h \geq \psi^2_h \} \). Suppose \( u^1_h \in K_1, u^2_h \in K_2 \) solve the following variational inequalities:

\[
a(u^1_h, v_h - u^1_h) \geq 0 \quad \text{for all } v_h \in K_1, \tag{3}
\]

\[
a(u^2_h, v_h - u^2_h) \geq 0 \quad \text{for all } v_h \in K_2. \tag{4}
\]

If \( \psi^1_h \leq \psi^2_h \), then

\[
u^1_h \leq u^2_h. \tag{5}
\]

**Theorem 3.** Suppose \( \Omega \subset \mathbb{R}^1 \) and the obstacle \( \psi \) is convex. Let \( G_{h_1}, G_{h_2} \) be two grids such that \( G_{h_2} \) is a refinement of \( G_{h_1} \), i.e., any node in \( G_{h_1} \) is also a node in \( G_{h_2} \). Denote by \( V_{h_1} \) and \( V_{h_2} \) the corresponding piecewise linear finite element spaces. Let \( K_{h_1} = \{ v_{h_1} \in V_{h_1} : v_{h_1} \geq \psi_{h_1} = \pi_{h_1}(\psi) \} \) and \( K_{h_2} = \{ v_{h_2} \in V_{h_2} : v_{h_2} \geq \psi_{h_2} = \pi_{h_2}(\psi) \} \). The discrete approximations \( u_{h_1} \in V_{h_1} \) and \( u_{h_2} \in V_{h_2} \) are the solutions of the following variational inequalities:

\[
a(u_{h_1}, v_{h_1} - u_{h_1}) \geq 0 \quad \text{for all } v_{h_1} \in K_{h_1}, \tag{6}
\]

\[
a(u_{h_2}, v_{h_2} - u_{h_2}) \geq 0 \quad \text{for all } v_{h_2} \in K_{h_2}. \tag{7}
\]

Then

\[
u_{h_1} \leq u_{h_2}. \tag{8}
\]

**Proof.** Let \( K_{h_2}^* = \{ v_{h_2} \in V_{h_2} : v_{h_2} \geq \psi_{h_1} \} \) and \( u_{h_2}^* \in K_{h_2}^* \) be the solution of the following variational inequality:

\[
a(u_{h_2}^*, v_{h_2} - u_{h_2}^*) \geq 0 \quad \text{for all } v_{h_2} \in K_{h_2}^*, \tag{9}
\]

which is equivalent to

\[
J(u_{h_2}^*) = \min_{v_{h_2} \in K_{h_2}^*} J(v_{h_2}). \tag{10}
\]
where
\[ J(v) = \int_{\Omega} \left( \frac{dv(x)}{dx} \right)^2 dx. \]

Then (3) is equivalent to
\[ J(u_{h_1}) = \min_{v_{h_1} \in K_{h_1}} J(v_{h_1}). \]

Define \( \tilde{u}_{h_2}^* \in V_{h_1} \) as the interpolant of \( u_{h_2}^* \) on the \( G_{h_1} \) nodes. Then \( \tilde{u}_{h_2}^* \in K_{h_1} \), which, together with (6), gives
\[ J(u_{h_1}) \leq J(\tilde{u}_{h_2}^*). \]

On the other hand, Lemma 1 gives
\[ J(\tilde{u}_{h_2}^*) \leq J(u_{h_2}^*). \]

By (8) and (9), we have
\[ J(u_{h_1}) \leq J(u_{h_2}^*). \]

Using the fact \( K_{h_1} \subset K_{h_2}^* \), (6) and (7), we have
\[ J(u_{h_2}^*) \leq J(u_{h_1}). \]

Then (10) and (11) give
\[ J(u_{h_1}) = J(u_{h_2}^*). \]

which, together with the uniqueness of the solution of (6), gives
\[ u_{h_1} = u_{h_2}^*. \] (12)

The convexity of \( \Psi \) gives \( \Psi_{h_1} \leq \Psi_{h_2} \). Applying the discrete monotonicity principle Theorem 2 to variational inequalities (5) and (4), we obtain
\[ u_{h_2}^* \leq u_{h_1}, \]

which, together with (12), gives
\[ u_{h_1} \leq u_{h_2}. \]

Consider a sequence of mesh refinement \( G_{h_1}, G_{h_2}, G_{h_3}, \ldots \) of \( \Omega \subset R^1 \), and the corresponding sequence of piecewise linear finite element approximations \( u_{h_1}, u_{h_2}, u_{h_3}, \ldots \) to the continuous solution \( u \). Then we have \( u_{h_1} \leq u_{h_2} \leq u_{h_3} \leq \cdots \leq u \) and \( \lim_{k \to \infty} \| u_{h_k} - u \|_{\infty} = 0 \). The \( L^\infty \)-error estimate were obtained in [6,1].

3. Two-dimensional case

First we demonstrate that Lemma 1 will not hold for 2D obstacle problems. Consider the right triangle \( O(0,0), A(1,0), B(0,1) \) and take \( u(x, y) = y \). Now we subdivide \( OAB \) into four smaller triangles by connecting the middle points on edges; see Fig. 1. We consider a family of piecewise linear functions \( v(x, y) \) on the refined mesh such that \( v|_{OA} = u|_{OA} = 0 \) and \( v(B) = u(B) = 1 \). It can be shown by direct computation that \( v(x, y) \) minimizes the energy \( \int_{OAB} |\nabla v|^2 dx \) if \( v(E) = \frac{1}{2} \) and \( v(D) = \frac{1}{2} \). We still denote the minimizer as \( v \). Then the conclusion of Lemma 1 will not hold for this example. Furthermore, we have \( v < u \), i.e. \( v \) is bent downward from \( u \).

Now we show that the monotone convergence theorem Theorem 3 will not hold for the two-dimensional case. Let \( \Omega = [-1, 1] \times [-1, 1] \) be a rectangular domain. Consider two grids \( G_{h_1} \) and \( G_{h_2} \) as shown in Fig. 2, where \( G_{h_2} \) is generated by connecting middle points of edges in \( G_{h_1} \). Suppose the obstacle \( \Psi \) is a thin round stick (impulse shape) near the center of the rectangle with \( \Psi(0, 0) = 1 \). We consider two discrete obstacle problems as defined in Theorem 3. Due to the symmetry of the problem, we can restrict attention to the piecewise finite element solution on the triangle \( O\tilde{Q} \). Let \( u_{h_1} \) be a continuous piecewise linear function on the grid \( G_{h_1} \), such that \( u_{h_1}(0,0) = 1 \), \( u_{h_1}|_{\partial \Omega} = 0 \), i.e., \( u_{h_1}|_{O\tilde{Q}} \) has the same shape as \( u \), and \( u_{h_2} \) be a continuous piecewise linear function on the grid
Fig. 1. A right triangular mesh.

Fig. 2. A two-level mesh refinement. On the left is the coarse grid $G_{h_1}$. On the right is the fine grid $G_{h_2}$.

$G_{h_2}$ such that $u_{h_2}(0, 0) = 1$, $u_{h_2}|_{\partial \Omega} = 0$, and $u_{h_2}|_{OPQ}$ has the same shape as $v$, where $u$ and $v$ are defined in the first paragraph. If $\Psi$ is thin enough, then $u_{h_1}$ and $u_{h_2}$ will stay above discrete obstacles $\Psi_{h_1} = \pi_{h_1}(\Psi)$ and $\Psi_{h_2} = \pi_{h_2}(\Psi)$. Then $u_{h_1}$ is the discrete solution for the obstacle problem on $G_{h_1}$ while $u_{h_2}$ is the discrete solution on $G_{h_2}$. The convexity of $\Psi$ gives $\Psi_{h_1} \leq \Psi_{h_2}$. But $u_{h_1} > u_{h_2}$ since $u > v$. Therefore Theorem 3 cannot be extended to the two-dimensional case.

Uncited references

References

