LES Simulations of Turbulent Combustion in a Type Ia Supernovae

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Abstract

We propose a 2D axisymmetric model of a type Ia supernova explosion, based on a front tracking sharp flame model. The calculation is free from adjustable turbulent transport parameters, and in this sense it is in the spirit of Large Eddy Simulation (LES) turbulence simulations. Since the mixing is dominated by the largest eddies, we resolve these and not the smaller ones. We believe this method results in a tolerable error, which, in any case understates the success of the explosion. We report successful explosions. Both the 2D and the LES nature of the model greatly simplify parameter identification. The 2D model allows multiple simulations and an exploration of unknown parameters, while the LES model removes parameters from the simulation.
1 Introduction

In this paper, we study turbulent combustion in Type Ia supernovae (SNIa). These supernovae - the brightest and most frequently observed - are believed to be thermonuclear explosions of white dwarfs. SNIa are important sources of energy and chemical elements deposited into the interstellar medium, and they are important distance indicators to measure the expansion of the universe. The chemical composition, the density and velocity of the ejecta, and the strength and brightness of the burning are all determined by the speed of thermonuclear burning [27]. The problem of turbulent combustion may also be important for other astrophysical objects such as novae, X-ray bursters, collapsing white dwarfs, etc. To determine the speed of turbulent combustion from first principles is of considerable interest.

Despite more than thirty years of intensive investigation, the physics whereby the carbon-oxygen core of a star near the Chandrasekhar mass explodes as Type Ia supernova is still debated. This indeed is a difficult problem because the nuclear flame propagates in an extensive medium in which gravity plays a role and several instabilities, Rayleigh-Taylor (RT), Landau-Darrieus (LD), and Kelvin-Helmholtz (KM) [16, 9], have time to develop over a large range of length scales. The length scales range from the thickness of the laminar flame, which is about $10^{-5}$ - 1 cm, depending on density [15], to the white dwarf radius of about $10^9$ cm. The thickness properties of the flame were investigated by [26] by equating the diffusion to the nuclear burning time scale. Thus the flame thickness, $\delta$, is given by $\delta \approx \sqrt{\lambda c E / S}$. Here $\lambda$ is the electron mean free path, $E$ is a characteristic energy per unit mass, $c$ is the sound speed, and $S$ is a typical energy generation rate. The upper bound on the flame thickness, 1 cm, occurs at a low density where we have applied a flame extinction model, and as we observe below, after which there is negligible burning. This thickness should be compared to our finely gridded simulations with a mesh spacing of about 2 km.

Our understanding of SNIa explosions are far from complete. To predict the explosion outcome, one needs to model the propagation of thermonuclear burning inside the exploding star. The mechanism and the speed of thermonuclear burning in SNIa remain an unsolved theoretical problem. A recent review by Niemeyer and Hillebrandt [17] contains a large list of relevant publications.

Owing to the vast range of relevant length scales involved, from millime-
ters to thousands of kilometers, it is not possible to resolve the full SNIa explosion in any multidimensional simulation. Thus attempts have been made to overcome this difficulty by classifying the problem into large scale calculations (LSC) and small scale calculations (SSC) [18, 24, 25]. LSC is used to study the large-scale perturbation of the flame in a supernova simulation, i.e., on scales of the radius of the exploding white dwarf. SSC is used to study the flame dynamics on smaller scales, for example to determine a laminar flame speed and to study the initial growth of instabilities. It has been demonstrated numerically that instabilities on smaller scales are smoothed by the propagation of the burning front into the unburned matter and are insufficient to produce SNIa observations [26, 9, 13, 16]. Here we propose to study the numerical simulations using LSC.

In a supernova simulation, the initial flame interface becomes highly unstable because the density of the ash is less than that of the unburned fuel and because gravity points in the direction of the less dense matter, giving rise to RT-instability. RT-unstable bubbles of hot ash rise through the cold unburned gas. This leads to shear-induced turbulence at the flame interface. The turbulence distorts and wrinkles the flame, thereby leading to an increasing surface area and energy generation of the flame front. Only a small part of the large flame front can be resolved computationally; a large region of the scale space or the flame's microstructure is generally modeled using numerical turbulence models. As will be evident from Section 2, most of the numerical studies of supernova explosion are performed using turbulence models. Thus the flame speed in these models becomes a function of the laminar and turbulent speeds. This turbulent velocity cannot be calculated from first principles [17]. In a one-dimensional simulation, the turbulent velocity was introduced as a free parameter, and its value was determined by fitting light curves and spectra [21]. In two-dimensional studies, various subgrid models for turbulence were used [17, 12, 22].

Our numerical models are free of any turbulence related parameters. Thus, in this sense we are proposing a LES type model for turbulent combustion. Large Eddy Simulations seek to solve directly enough of the large spatial scales. Thus our flame is advanced normally using a laminar flame speed only. The scientific justification for the LES approach, i.e., that the small turbulent scales not modeled will not greatly affect the answer, lies in a mesh convergence study. We postpone this issue to a later paper, and simply report that preliminary results suggest converged total combustion rate with a few more orders of mesh refinement, and that this convergence will
not greatly change the results reported here. Besides the initial composition, the central density and the location of the flame front and its perturbations determine the simulation results.

In order to model turbulent thermonuclear combustion accurately, we track the flame front. The reasons for this are simple. First, it is important to reproduce the geometry of the flame front as accurately as possible, because its effective area also determines the rate of fuel consumption. Secondly, since nuclear reactions are very sensitive to the temperature \( T \) and their rates depend on a very high power of \( T \), mixing fuel and ashes numerically in certain mesh points on both sides of the flame front and smearing out the temperature gradient will lead to modeling difficulties for (local) energy generation rates. Therefore, if in an algorithm one wants to follow the nuclear transmutations explicitly, one is forced to guarantee that temperature jumps due to fast reactions are tracked or modeled with high precision. Here we propose to conduct numerical simulations for Type Ia supernova by applying the flame tracking algorithm which will be summarized in Section 3. Before we proceed to describe our model, we discuss briefly some of the important numerical studies of SNIa that have been obtained by others.

2 Previous Work

Khokhlov [9] studied the thermonuclear, carbon-oxygen deflagration flame in supernovae using a flame capturing technique. His simulations showed that the turbulent flame speed, \( D_t \), is less than that of the intensity of the turbulent motion, \( U' \). He found out numerically that the theoretical relationship between \( D_t \) and \( U' \), \( D_t \sim \sqrt{D_t U'} \), holds also for a thermonuclear flame; here \( D_t \) is the laminar flame speed. Using this relation and simple scaling arguments, Khokhlov estimated that the turbulent speed reaches a maximum of 6 percent of sound speed. That was too small for the deflagration mechanism of supernova since the flame has to accelerate to a speed comparable to the sound speed in order to cause an explosion. Thus he suggested that RT instability was not sufficient to produce an energetic Type Ia supernova explosion and that a delayed detonation mechanism was required for a successful explosion. Other workers have proposed deflagration models for SNIa explosions.

Livne [13] examined the propagation of the flame using a two-dimensional implicit/explicit hydrodynamic code, Vulcan, and found out that the large-
scale perturbations of the flame, though they grew significantly, were not sufficient to produce an acceptable model for SNIa’s. The large scales of the instability were captured by the grid while the scales that were smaller than the grid resolution were modeled by a mixing-length approximation. The flame speed, \( v_f \), was defined as \( v_f = \max(v_c, v_t) \), where \( v_c \) is the conductive flame speed and \( v_t \) is the turbulent flame speed. The turbulent flame speed is approximated numerically as \( v_t = \sqrt{\Phi \cdot D^\perp \frac{\delta \rho}{2\rho}} \), where \( \Phi \) is the gravitational potential, \( D^\perp \) is a vector having the length of the cell face and direction normal to the face; \( \delta \rho \) is the density difference between the burned and unburned stellar matter, \( \rho \) is the unburned density. This and a number of other turbulence models are motivated by the Froude number of terminal rise velocity of a periodic array of bubbles with a subgrid radius set by the mesh spacing. Thus his simulations used a turbulent model, and produced about \( 1.4 \times 10^{50} \) ergs of energy when the flame was not perturbed and about \( 4 \times 10^{50} \) ergs in the case of perturbed simulations. These energies were not enough to produce a supernova explosion. All his simulations were performed using a centrally ignited flame front.

Ropke et al. [23] continued their numerical simulation of the flame in the distributed regime. All the earlier numerical simulations of SNIa were done in the flamelet regime. The flamelet regime is the burning regime of “large scale turbulence” where the flame is deformed by interaction with turbulent motions. The small scale turbulence regime where turbulent eddies actually penetrate the internal flame structure is known as distributed regime.

To model the distributed regime, they used an assumption that at densities below \( 10^7 \) g/cc, the flame enters into the distributed burning regime. In this regime the turbulent velocity was derived using a formula \( \sqrt{s_1 \nu l / l_f} \), where \( s_1 \) is the laminar flame speed, \( \nu \) is the turbulent velocity fluctuation, \( l_f \) is the flame thickness, and \( l \) is a certain length scale. The \( \nu \) and \( l \) were derived using a sub-grid scale model. \( s_1 \) and \( l_f \) are provided by fits to the data given by Bell et al [2]. By continuing the burning in the distributed region they found it produced about 17 percent more energy compared to the simulations where burning was not continued. The amount of energy produced was about \( 5 \times 10^{50} \) ergs. Also, the model gave better results in the sense that almost no unburned materials were left in the central core, expansion velocities of the ejecta were high and more intermediate mass elements were produced. All these were problems in the previous simulations).

Muller et al. [14] studied the evolution of the carbon core beginning at
the runaway stage using an explicit two-dimensional Eulerian hydrocode and a simplified reaction, \( C \rightarrow Ni + q \). The \( q \) used was \( 7 \times 10^{17} \text{ ergs/gm} \). They did several numerical simulations and their studies showed that the initially spherically symmetric burning front is RT-unstable. While discussing the propagation of the burning front, the authors noted that the propagation of the burning front could be erroneously enhanced by numerical diffusion or underestimated because the burning could be turbulent. The interplay of advection, Eulerian mixing, numerical diffusion and finiteness of the grid is responsible for when and how the instability of the flame front will occur. Also, they claimed that their numerical experiments showed a variety of possible evolutionary paths for the degenerate carbon-oxygen cores, ranging from spherical detonation to non-spherical deflagration.

Niemeyer et al. [19] studied a series of two-dimensional numerical simulations using an Eulerian PPM-based code. In all the computations, the flame had been smeared out over a large region by turbulent diffusion. The effective macroscopic burning speed was found using the maximum of all competing transport velocities at the grid: \( u(\Delta) = \max[u_{\text{lam}}, v(\Delta), v_{\text{r}}(\Delta)] \), where \( u_{\text{lam}} \) is the laminar flame speed, \( v_{\text{r}}(\Delta) \) is the asymptotic velocity of rising buoyant bubbles with a radius \( \Delta \), \( \Delta \) is the grid size and \( v(\Delta) \) is the turbulent fluctuation velocity obtained from the kinetic sub-grid energy. In this paper the authors concluded that: 1. to produce a successful explosion in the first deflagration phase, flame ignition must occur at number of points, 2. the simulation involving a centrally located flame produced only a weak Type Ia supernova, barely producing the required nuclear-burning products necessary to account for the light curves and and kinetic energy of the spectra, 3. simulations using multiple-point ignitions were possible candidates for some Type Ia supernova, although the amount of \( Ni^{56} \) produced was not enough to account for the observed light curves, and thus one could summarize that the deflagration model of Type Ia supernova with small number of burning bubbles was not likely to produce a powerful Type Ia supernovae, 4. the off-center, single-point ignition burned only \( 0.3M_\odot \) of the fuel, and thus could not produce a supernova explosion, 5. in the overall comparison of the expected nucleosynthesis, all off-center models were closer to the expectations from Type Ia supernova observations and solar abundance studies than that of the centrally-ignited model, and 6. deflagration-to-detonation transition did not occur as the turbulent velocity of the burning front remained very subsonic.

Most of the previous two-dimensional models of supernova simulation
failed in the sense that an inadequate amount of burning occurred to produce a credible explosion [14, 10, 17, 1]. Thus with the advent of increased computer power, many embarked on supernova simulations in three dimensions to test whether it was the deficiency of a two-dimensional studies that they failed to produce a powerful explosion. In 3D more surface area of the star will be burned producing more energy and thus a successful explosion. Khoklov [11] presented the first results modeling a 3D deflagration explosion in a non-rotating, Chandrasekhar-mass carbon-oxygen white dwarf. In his model, the flame was modeled using a reactive-diffusive method and advanced using a flame-capturing technique. The fluid dynamic equations were solved using an explicit, second-order, Godunov-type, adaptive mesh refinement algorithm ALLA. The laminar speed was determined by the competition between the heat diffusion and nuclear energy generation inside the front. To describe burning on scales that are not resolved, the flame propagated with a speed $\mathcal{S}$, where $\mathcal{S} = \max(S_1, S_{\text{sub}})$, with $S_{\text{sub}} = 0.5 \sqrt{2A \alpha \Delta r \langle s_1 | g \rangle + s_2 \max(0, \mathbf{n} \cdot \mathbf{g})}$. Here $\Delta r$ is the computational cell-size, $\mathbf{n}$ is the unit vector normal to the flame, $s_1$ and $s_2$ are variables that are either 1 or 0, $A$ is the Atwood number, and $\alpha$ determines the driving scale and was taken as approximately 1. $S_1$ is the laminar flame speed. The computation was performed in one octant. Burning was initiated in a small sphere of radius $3 \times 10^6$ cm around the origin. Khoklov found that about 60 percent of the white dwarf was burned by the end of the simulation. Burning continued in the central parts of the star. Thus he speculated that a few more units of $10^{50}$ ergs of energy would be produced before the flame was completely quenched, producing approximately $(1 - 1.3) \times 10^{51}$ ergs of kinetic energy. This was about 50 - 80 percent of the total energy required to produce a typical SNIa explosion. Through some parameter studies, the author found that by decreasing the initial carbon composition, less energy was released. Also a 30 percent deviation of the ignition site from the center was enough to cause a highly asymmetric flame. However, there were some features in the simulation model that lead the author to conclude that it was an incomplete model of a SNIa. One of these features was the presence of a massive, $\approx 0.4 M_\odot$, outer layer of unburned carbon-oxygen surrounding the burned layer. Presence of the unburned layer limits the maximum expansion velocities of the intermediate mass elements, and thus could not produce the observed spectra of SNIa. Another drawback of the model was that intermediate mass elements were produced throughout the white dwarf, presenting another difficulty for spectral modeling. Also, unburned carbon-oxygen was present in the central region of the star, and
that posed problem for modeling the SNIa spectra.

Reinecke et al. [22] did numerical simulations in three-dimensions using multiple-point ignitions. The flame velocity, \( s_u \), was modeled as \( s_u = \sqrt{2\epsilon_{\text{sg}}} \), where \( \epsilon_{\text{sg}} \) is the turbulent kinetic sub-grid energy. The model was based on a level set method. Two different calculations were performed by the authors to investigate the off-center ignition model in three dimensions. In one of the simulations, five bubbles with a radius of \( 3 \times 10^6 \) cm were distributed randomly in the simulated octant within \( 1.6 \times 10^7 \) cm of the star’s center. The second simulation contained nine bubbles of radius \( 2 \times 10^6 \) cm within \( 1.6 \times 10^7 \) cm of the white dwarf’s center. The first simulation was done on a grid of \( 256^3 \) cells, while the second simulation used a grid size of \( 512^3 \) cells. The first simulation produced \( 9.47 \times 10^{50} \) ergs of energy while the second simulation produced \( 11.26 \times 10^{50} \) ergs. Thus the models were successful in the sense that the explosion energy and also the average chemical composition of the ejecta seemed to fit the observed data of SNIa.

All of the above multidimensional studies of supernova explosions are based on turbulence models. All the two-dimensional models shared a common fate - an inadequate amount of burning occurred to produce a credible explosions; thus three-dimensional studies were required to produce good supernova explosions. It becomes computationally unrealistic to use three-dimensional models to do parameter studies. Also some authors [22, 19] used off-center and multiple-point ignition models to produce good results. In all these respects, our two-dimensional spherical axisymmetric model is different because we do not use a turbulence model for the flame speed. Thus no hydro parameters related to turbulence are used to adjust the flame speed. We are able to produce good burning and hence a good explosion. We do not use multiple-point ignition. The flame is initially positioned at the center of the white dwarf. By explicitly tracking the front, we have been able to eliminate numerical diffusion across the tracked front and thus artificial enhancement of the flame velocity. Also it is computationally feasible to do parameter studies using a two-dimensional model.

In the next few sections, we present our numerical model, algorithm and results in detail.
3 Formulation

The relations governing the transition from unburnt to burnt gas in combustion are derived from the three conservation laws of mass, momentum, and energy in the same way as are the relations governing a shock transition. The only difference is that the total specific energy equations contained an extra term $q$, the chemical or nuclear binding energy, $E = e + \frac{1}{2} \vec{v} \cdot \vec{v} + q$ with specific internal energy $e$ and velocity $\vec{v}$.

To study the flow in a spherical/cylindrical axisymmetric geometry, we transform the rectilinear coordinate system into a spherical/cylindrical coordinate system, and get the following equations, where $\nabla = \frac{\partial}{\partial r}$ or $\nabla = \frac{\partial}{\partial r} + \frac{\partial}{\partial z}$ for spherical and cylindrical coordinate systems respectively.

$$\rho_t + \nabla \cdot (\rho \vec{v}) + \frac{\alpha \rho u}{r} = 0,$$
$$\frac{\partial}{\partial t} (\rho \vec{v}) + \nabla \cdot (\rho \vec{v} \otimes \vec{v}) + \nabla p + \frac{\alpha \rho u \vec{v}}{r} = \rho \vec{b},$$
$$\frac{\partial}{\partial t} (\rho E) + \nabla \cdot \rho \vec{v} (E + p/\rho) + \frac{\alpha \rho u (E + p/\rho)}{r} = \rho \vec{v} \cdot \vec{b},$$

where $\rho$ is the mass density, $\vec{v}$ is the fluid velocity, $p$ is the thermodynamic pressure, $\vec{b}$ is the body force per unit mass, $E = e + \frac{1}{2} \vec{v}^2$ is the total energy and $e$ is the specific internal energy. The thermodynamic variables $\rho$, $p$ and $e$ are related by the equation of state $p = p(\rho, e)$. The first equation is the continuity equation, the second is the conservation of momentum equation, and the third is the differential form of the energy equation.

We have assumed a gamma law gas state with $\gamma = 4/3$. For a 2D axisymmetric flow $\alpha = 1$ while $\alpha = 2$ for a 1D spherical symmetry; $u$ is the radial component of the velocity; $r$ is the distance of a point from the origin for the spherical symmetric case while $r$ is the distance of a point from the $z$ axis for the cylindrical symmetric case.

The above equations, along with the following Rankine-Hugoniot equations,

$$\left(\rho v_N\right)_{\text{jump}} = s_f(\rho)_{\text{jump}},$$
$$\left(\rho u_N^2 + p\right)_{\text{jump}} = s_f(\rho v_N)_{\text{jump}},$$
$$\left(\rho u_N v_T\right)_{\text{jump}} = s_f(\rho v_T)_{\text{jump}},$$
$$\left(\rho E v_N + \rho v_N \vec{v}\right)_{\text{jump}} + \left(\rho q v_N\right)_{\text{jump}} = s_f(\rho E)_{\text{jump}} + s_f(\rho q)_{\text{jump}},$$

where $v_N$, $v_T$ are the normal and tangential velocity respectively, $(x)_{\text{jump}} = \ldots$.
\( x_b - x_u, \) \( x_b \) and \( x_u \) are the burnt and unburnt states respectively, and \( s_f \) is the laminar flame speed.

4 Front Tracking Algorithm

The numerical study uses a front tracking method, as implemented in the code \textit{FronTier}. Front tracking is an adaptive computational method in which a lower dimensional moving grid is fit to and follows the dynamical evolution of distinguished waves in a fluid flow. The lower dimensional grid is embedded in a higher dimensional fixed rectangular grid. Detailed description of the code can be found in [7, 5, 3, 6, 8]. However, a brief description of the code is as follows. In \textit{FronTier}, each time step is divided into two processes: propagation of the front and updating the solution on the spatial grid. The front is the material interface that separates fluids of distinct densities; the evolution of the front, over time, can be tracked. During propagation of the front, we project the front dynamics into the normal and the tangential directions, which are thus split into two one-dimensional problems. The problem in the normal direction is a non-local Riemann problem with an idealized jump discontinuity. After updating the states on each side of the front or contact discontinuity, the front is propagated by moving each front point using the computed wave speed. We then solve the problem in the tangential direction by updating the states along each side of the front using MUSCL (Monotone Upwind Schemes for Conservation Laws) schemes. Finally all the states on any off-front spatial grid region are updated again by using the MUSCL schemes. In this way, we never perform any finite differencing across the front; also, we keep all the discontinuities perfectly sharp by tracking any discontinuous wave. Thus the main advantage of the Front Tracking method is that it eliminates numerical diffusion at tracked points that would otherwise occur in a standard finite-difference method.

5 Numerical Simulation

In this section, we present the results of our numerical simulations to study the amount of energy released by thermonuclear burning of SNIa. This study is possible because most of the energy is released during the first second of a supernova explosion. Most of our simulations used a 500\(^2\) mesh (6 \times 10^5
cm mesh spacing). A preliminary mesh convergence study is shown in Fig. 1 and Table 1.

![Figure 1: Mesh refinement study using 200², 400², 800² and 1600² grids.](image)

<table>
<thead>
<tr>
<th>( \Delta{x} \times 10^6 \text{ cm} )</th>
<th>Mass Burned ( \times 10^{-3} \text{ gm} ) (% burned)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.5 (17.6%)</td>
</tr>
<tr>
<td>0.75</td>
<td>1.06 (36.9%)</td>
</tr>
<tr>
<td>0.375</td>
<td>1.6 (55.7%)</td>
</tr>
<tr>
<td>0.1875</td>
<td>2.3 (80.1%)</td>
</tr>
</tbody>
</table>

Table 1: Dependence of mass burned on grid refinement.

We consider a simplified reaction by combining all the slow, fast and intermediate nuclear reaction processes into a single effective reaction: \( ^{12}\text{C} \rightarrow 3N^{56} + q \). This yields a specific energy release of \( q = 9.28667 \times 10^{17}\text{ ergs/g} \) [25]. Many papers assume that the energy release for this equation is \( q = 7 \times 10^{17}\text{ ergs/g} \) [13, 14]. We are unable to confirm which of these values is correct and thus present comparative results of simulations using both values of \( q \) in Fig. 2. We notice that more mass is burned but marginally less energy is released for the simulation with the lower energy release value \( q = 7 \times 10^{17} \text{ ergs/g} \).

In our model, the flame propagates in a normal direction with a prescribed laminar flame speed. For the laminar flame speed, an approximate
Figure 2: Comparison of two energy release rate values. Flame front plots for $q = 7 \times 10^{17}$ ergs/g (top left) and $q = 9.28 \times 10^{17}$ ergs/g (top right). Below the total mass burned and energy released are compared for the two values of $q$.

function was derived by Timmes and Woosley [26] by performing fully resolved, microscopic combustion simulations in white dwarf matter for many different parameter sets. Thus for this paper, we use Timmes and Woosley’s derived laminar speed:

$$s_t = 9.2 \times 10^6 \text{cm/s} \left(\frac{\rho}{2 \times 10^9 \text{g/cm}^3}\right)^{0.805} \left(2 \times X(C)\right)^{0.889},$$

where $\rho$ is the density of the unburned mass, and $X(C)$ is the carbon mass fraction. The above expression is used throughout our simulations, and is a lower bound for the turbulent speed, which is not used in our model.

The thermonuclear burning for the simulations take place at fuel densities above $10^7$ g/cc. We carried out simulations to determine the dependence of the mass burned on a flame extinction model threshold, which terminates
burning whenever the density of the burning material fell below a certain threshold. Three simulations were compared, one with no flame extinction model, one with a flame extinction threshold set at $10^6 \, \text{g/cc}$ and one with the extinction threshold set at the transition to the distributed flame regime of $10^7 \, \text{g/cc}$. Stellar expansion quenches nuclear reactions and hence mass burning at densities less than $\approx 10^6 \, \text{g/cc}$ [12]. At densities below $\approx 10^7 \, \text{g/cc}$, turbulence disrupts the flame sheet and the burning enters a distributed flame regime [20]. The results are given in Fig. 3, and we find that there is no visible difference among these cases. We conclude that the extinction threshold is not important to our conclusions. Analysis of our simulations reveals that the densities do not fall below either threshold. Thus the reason for the leveling of the burn curve in Fig. 3 lies in the density dependence of the flame speed (3) which in effect accomplishes its own extinction threshold.

![Flame quenching at two different thresholds is compared to a burn model with no quenching. The three burn curves, (solid, dashed and dotted), are indistinguishable.](image)

The computation was done in a square domain $[0, r_1] \times [0, z_1]$ with $z_1 = r_1 = 3 \times 10^8 \, \text{cm}$. The origin is denoted by $R_0 = (0, 0)$. Let $R$ denote the distance from any point in the computational domain to the origin. At $t = 0$, a circular region around the center of the star is ignited. The initial flame surface is located on a perturbed circle $R(\theta) = R_0 + \sum_{n=n_{\text{min}}}^{n_{\text{max}}} A_n \cos(4n\theta)$, where the azimuthal angle $\theta \in [0, \pi/2]$, and $n_{\text{min}} = 4$, $n_{\text{max}} = 8$ are the minimum and maximum frequencies respectively. The Fourier mode amplitudes $A_n$ are generated by Gaussian sampling. Computations were performed with
the mean radial distance $R_0$ of the initial flame front from the center of the star of approximately 150 km. In our experiment, the inner fluid is burned and is considered to be of pure nickel; the outer fluid is pure unburned carbon. Due to the rotational symmetry about the z-axis, we are considering an axisymmetric perturbed spherical exploding problem. Reflecting boundary conditions are used at the left and lower sides, i.e. along the $r = 0$ and $z = 0$ axes. Flow through boundary conditions are applied at the upper and right boundaries of the domain so that outbound waves will exit the domain.

We use self-gravity in a spherically axisymmetric approximation for our computation. In such a setup, the density $\rho = \rho(r, z)$ is a function of the cylindrical coordinates $r$ and $z$. The gravitational force points towards the origin with the magnitude $\frac{G M(R)}{R^2}$. $M(R) = 2 \int_{R'}^R \rho(R') \, dV$ is the mass of the star enclosed inside the shell of radius $R$. In order to compute this mass numerically, the computational domain is divided into $N$ spherical shells. For each shell, the shell mass is computed by adding up the contributions from cells inside that shell.

The initial central density is taken as $\rho_c = 2.9 \times 10^9 $g/cm$^3$. Starting from the central pressure $P(\rho_c)$, the equations of hydrostatic equilibrium, $dP/dR = -G M \rho / R^2$ and $dM/dR = 4 \pi \rho R^2$, were integrated outward. The integration was stopped at a density of $10^{-3}$g/cm$^3$; for larger radii, the density was kept constant at this value. Thus the outer region of the computational domain is not in hydrostatic equilibrium and matter starts to fall down onto the star's surface. However, this had very little impact on the explosion itself since the mass affected is very small. This procedure resulted in white dwarf mass of $2.87074 \times 10^{33}$g.

By the end of the simulation the central density has dropped by a factor of 7 to a value of $4.2 \times 10^8$g/cc, as can be seen in the left image of Fig. 4. The central temperature starts with a value of about $10^{11}$K and drops to a value of $5.3 \times 10^{10}$K by the end of the simulation (Fig. 4). We compare the value of our initial central temperature to the temperature $10^9$K which is required for immediate processing of carbon to nickel to take place [14].

The density evolution can be further understood by studying the Atwood number for the simulations. The density contrast, $1 - \frac{\rho_u}{\rho_b}$, between the burned and unburned material is $[0.25, 0.54]$ (Fig. 5). This value is close to the range of $[0.1, 0.5]$ mentioned in [15]. The Atwood number for the simulation is within a range of $[0.14, 0.37]$, as evident in the Fig. 5. Here the Atwood number is defined as $\frac{\rho_u - \rho_b}{\rho_u + \rho_b}$, where $\rho_u$ is the unburned density and $\rho_b$ is the
Figure 4: Density (left) and temperature (right), averaged over a spherical shell of radius $R$, at the beginning and end of the simulation. The spike in the temperature is an artifact of the initialization, whereby the initially burned nickel has the same temperature as the ambient unburned matter.

Figure 5: Left: Atwood number for the tip of the leading bubble as a function of time. Right: Density contrast between burned and unburned material at the tip of the leading bubble as a function of time.

We present a model based on the effective Atwood number [4]. For this purpose we introduce several definitions. Let $R_b$ be the radius of the largest bubble, i.e. the largest radius on the flame front; let $R_u$ be the smallest radius. In view of the moderate Atwood numbers for the SNIa flame front, we regard $R_u = (R_b + R_u) / 2$ as the radius of uniform expansion of the star. Radii $R > R_u$ lie the the bubble portion of the flame front and $R_b - R_u =$
Figure 6: Top left: plot of effective Atwood number \( A(t) \) and top right: effective gravity \( g(t) \) as a function of time. Bottom left: \( 2 \int \int A(t)g(t)dt \) versus \( R_{HT,b} \). Bottom right: comparison of \( R_b \) (light dotted), \( R_u \) (dash), and the height obtained by integrating the laminar flame speed (dark solid).

\( (R_b - R_u)/2 \) is the length to which the theory of bubble growth in the RT instability applies. We call this length \( R_{HT,b} \). The effective Atwood number \( A(t) \) is an Atwood number defined as the average of a local Atwood number \( A(R, t) \) over the outer half of the bubble region defined above, where \( A(R, t) \) is defined in terms of the average burned and unburned densities on a spherical shell of radius \( R \). A similar definition applies to the effective gravity \( g(t) \). Then a dimensionless self similar growth constant \( \alpha_b \) is defined as the slope of the plot of \( R_{HT,b} \) versus \( 2 \int \int A(t)g(t)dt \). The plot of \( R_{HT,b} \) versus \( 2 \int \int A(t)g(t)dt \) is given in bottom left image of Fig. 6, and from this we determine a slope \( \alpha_b \approx 0.07 \). We conclude that the simulation is consistent with accepted ideas governing RT mixing. The same figure also contains a comparison of \( R_b \), \( R_u \) and the height obtained by integrating the laminar flame speed (bottom right image). The laminar flame speed is given by
equation 3. As defined here, the RT instability height can be at most equal to the uniform expansion height, and we note that this is essentially the case. We notice that the radius of the uniform expansion of the star exceeds the height obtained by integrating the laminar flame speed from around time \( t \approx 0.35 \). Thus, we can conclude that from around this time the flame expansion has become independent of effects due to laminar flame speed directly and is expanding largely due to advection, and that advection is approximately evenly divided between RT instability growth and uniform expansion.

We compare the laminar speed, the advection speed and the model bubble speed to the motion of the burn front as a function of time in Fig. 7. We have also plotted the speed of the largest bubble and compared it with the speed of the bubble which we compute directly using the formula \( 2\alpha \int A(t)g(t)dt \) (Fig. 7). The value of \( \alpha \) used here is \( \alpha = 0.07 \).

![Figure 7](image_url)

*Figure 7: Left: laminar speed. Right: comparison of the advection speed (solid), the laminar speed (light dashed) for the tip of the largest bubble and the model speed \( 2\alpha \int A(t)g(t)dt \) (bold small dash).*

The surface area of the flame as a function of time is shown in Fig. 8. Comparison with Fig. 7 left shows that the laminar flame speed at the bubble tip has dropped to 10% or less of its initial values when the area begins its rapid growth. While we expect that the area, when growing rapidly, will be unresolved due to grid effects, we thus observe that the impact on the burn rate is not large.

We investigated the effect the amplitude of the perturbation on the energy generation rate. To do that, we carried out a series of numerical simulations. Below we present the results of our initial studies. The statistical quantities,
Figure 8: Surface area of the flame as a function of time.

i.e., the amount of burnt mass and energy produced are averaged over 5 realizations.

Fig. 9 shows the influence of amplitude by wavelength ratio on the simulation results. We consider three geometric parameters in the configuration of the initial perturbed interface: the average wavelength $\lambda_0$, the initial flame radius $R_0$, and the average amplitude $a_0$. If we treat the wave length $\lambda_0$ as the length scale, then we define $\tilde{a} = a_0/\lambda_0$ to be the dimensionless amplitude. Here initial wave length is calculated as $\lambda_0 = \pi R_0/2n_0$, where $n_0$ is the number of modes in the initial perturbation. Throughout this paper, the dimensionless amplitude is called as amplitude by wavelength ratio.

We performed two simulations using 0.24 and 0.12 as the initial amplitude by wavelength ratios. The initial flame was located at a mean distance of 150 kilometer from the origin. The initial radius of the star is $3 \times 10^8$ cm. Fig. 10 shows the front plots of three elements of this ensemble for one of these two simulations. All the simulations were performed using a uniform $500 \times 500$ grid. For the two simulations of 0.24, 0.12 dimensionless amplitude by wavelength ratio, $11.25 \times 10^{50}$ and $12.95 \times 10^{50}$ ergs of energy, respectively, is liberated. Those with larger amplitude by wavelength ratio produced less energy compared to those perturbed fronts with smaller amplitude by wavelength ratio. We have a healthy explosion for both the two cases.
Figure 9: *Comparison of total burned mass versus time, for simulations where the amplitude by wavelength ratio was varied.* For the simulation with 0.12 (0.24) ratio, 48.6 (42.2) percent mass was burned.

## 6 Conclusion

All the simulations produced healthy supernova explosion for the first time in two-dimensional studies. A number of numerical and physical modeling issues should be addressed before the estimates of burning are considered to be definitive:

- full reaction network;
- realistic equation of state;
- initial composition as a radially dependent mixture of carbon and oxygen;
- 3D effects;
- elimination of the spherical approximation in the gravity calculation;
- mesh convergence studies;
- a possible role for turbulence models in the distributed flame regime.

We also intend to study the dependence of the simulations on the number of initial modes. Finally, we notice that the initial radius, and initial amplitude
to wavelength ratio are sensitive parameters, and deserve further study. We hope to address most of these issues in future studies.

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References


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Figure 10: Flame front for a perturbed interface with initial amplitude by wavelength ratio of 0.12. The image shows three simulations from an ensemble of five. The initial flame had an average initial radius $0.15 \times 10^6$ cm. The flame front at time $t = 0$ was initialized using five Fourier modes. The first image in each row shows the flame front at time $t = 0$, the second image at the end of the simulation.