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NUMERICAL SOLUTION OF VARIATIONAL INEQUALITIES

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ABSTRACT

If one wants to minimize a nonlinear functional it is often fruitful to consider the relationships which must hold at a minimum. If the functional is differentiable and the permitted variations at the minimum constitute a linear space this process gives equations that the minimum must satisfy, and if the functional is quadratic these equations are linear. However, if the set of permitted variations is constrained (for example to nonnegative functions) or the functional is nondifferentiable, then one may find inequalities instead of equations.

We are interested in numerically approximating solutions of two types of variational inequalities. The first one is variational inequalities with constrained admissible set, frequently called obstacle problems. The second type is variational inequalities with a non-differentiable term. An important example of this type is rigid visco-plastic Bingham fluid.

L^∞ -error estimates for numerical solutions of obstacle problems have been investigated by C. Baiocchi and J. Nitsche. Though Nitsche's estimate is optimal ($O(h^2|\ln h|)$), the discrete solution he defined is not in general computable because the obstacle itself is not discretized. A new monotonicity principle for a discrete obstacle problems is applied to obtain an optimal L^∞ -error estimate for an approximation in which the obstacle is only respected at the vertices of the triangulation. This result both uses and improves the Nitsche's estimate.

Numerical computation of Bingham fluid flow has been studied by M. Fortin and R. Glowinski, but error estimates are not yet available for their methods. A new numerical method for approximate solution of time-dependent flow of Bingham fluid in cylindrical pipes which uses regularization of nondifferentiable term is studied. Error estimates are described for the case in which the discretization is done using piecewise linear finite elements in space and backward differencing in time.

CHAPTER 1

A MONOTONICITY PRINCIPLE AND L^∞ -ERROR BOUND FOR A DISCRETE OBSTACLE PROBLEM

1.1 Introduction

L^∞ -error estimates for numerical solutions of obstacle problems have been investigated by Baiocchi [1] and Nitsche [16]. Though Nitsche's estimate is optimal ($O(h^2|\ln h|)$), the discrete solution he defined is not in general computable because the obstacle itself is not discretized. Here we give an optimal L^∞ -error estimate for an approximation in which the obstacle is only respected at the vertices of the triangulation. The proof uses Nitsche's result and a monotonicity property of the discrete problem.

The problem we want to approximate can be described in several different ways, the formulation in terms of a variational inequality convenient here. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with boundary Γ . Define $a(v, w)$ to be the Dirichlet form:

$$a(v, w) = \int_{\Omega} \nabla v \cdot \nabla w dx, \quad v, w \in H_0^1(\Omega).$$

The obstacle is a continuous function $\Psi : \bar{\Omega} \rightarrow \mathbb{R}$ with $\Psi|_{\Gamma} < 0$. Set $K = \{v \in H_0^1(\Omega) : v \geq \Psi \text{ in } \Omega\}$. The obstacle problem is, given $f \in L^2(\Omega)$, find $u \in K$ such that

$$a(u, v - u) \geq (f, v - u) \quad \text{for all } v \in K, \tag{1.1}$$

where (\cdot, \cdot) is the L^2 inner product. The basic properties of the solution, including existence and uniqueness, were established by Lions and Stampacchia [15]. A collection of physical applications of this problem and related numerical methods were discussed in a book by Glowinski [10].

1.2 Discrete Monotonicity Principle

Suppose henceforth that the boundary of Ω , Γ , is polygonal. For a triangulation T of Ω , let $h = h(T)$ be the max of the lengths of the edges. We say that a triangulation satisfies the maximum angle condition if no triangle in it has an angle that exceeds $\pi/2$. We say that a family of triangulations satisfy the maximum angle condition if each member does.

Given a triangulation T_h , let $\tilde{V}_h = \tilde{V}_h(T_h)$ denote the space of continuous piecewise linear functions over T_h . Take $V_h = \tilde{V}_h \cap H_0^1(\Omega)$. For $v \in C^0(\bar{\Omega})$ let $\pi_h(v) \in \tilde{V}_h$ be the nodal interpolant of v ; $v = \pi_h(v)$ at each vertex. Let $\Psi_h = \pi_h(\Psi)$ define the discrete obstacle, and take $K_h = \{v_h \in V_h : v_h \geq \Psi_h\}$. The discrete approximation of u is given by $u_h \in K_h$, such that

$$a(u_h, v_h - u_h) \geq (f, v_h - u_h) \quad \text{for all } v_h \in K_h. \quad (1.2)$$

Note that the set K_h of admissible functions is defined by a finite set of inequalities.

First we introduce some notation. We denote $\sup(v_h, w_h) \in V_h$ the least upper bound and $\inf(v_h, w_h) \in V_h$ the greatest lower bound for any $v_h, w_h \in V_h$; these are just the node-wise max and min of the two functions. Further if we introduce the notation $v_h^+ = \sup(v_h, 0)$, then we have identities $\sup(v_h, w_h) = v_h + (w_h - v_h)^+$, and $\inf(v_h, w_h) = w_h - (w_h - v_h)^+$. Now we can prove a useful theorem. The idea of the proof is motivated by Brezis and Stampacchia [4], Haugazeau [12].

Theorem 1.1 *Suppose the triangulation T_h satisfies the maximal angle condition. Suppose that K_1, K_2 are two convex subsets of $V_h = V_h(T_h)$ and that $u_1 \in K_1, u_2 \in K_2$ solve the following variational inequalities:*

$$a(u_1, v_h - u_1) \geq (f, v_h - u_1) \quad \text{for all } v_h \in K_1, \quad (1.3)$$

$$a(u_2, v_h - u_2) \geq (f, v_h - u_2) \quad \text{for all } v_h \in K_2. \quad (1.4)$$

Suppose also that K_1, K_2 satisfy the following monotonicity property: For all $v_1 \in K_1, v_2 \in K_2, \inf(v_1, v_2) \in K_1, \sup(v_1, v_2) \in K_2$. Then

$$u_1 \leq u_2.$$

Proof. With $v_h = \sup(u_2, u_1) = u_2 + (u_1 - u_2)^+ \in K_2$ in (1.4), we have

$$a(u_2, (u_1 - u_2)^+) \geq (f, (u_1 - u_2)^+). \quad (1.5)$$

Taking $v_h = \inf(u_2, u_1) = u_1 - (u_1 - u_2)^+ \in K_1$ in (1.3) gives

$$a(u_1, -(u_1 - u_2)^+) \geq -(f, (u_1 - u_2)^+). \quad (1.6)$$

By addition of (1.5) and (1.6), we have

$$a(u_1 - u_2, (u_1 - u_2)^+) \leq 0. \quad (1.7)$$

To complete the proof we will show that $(u_1 - u_2)^+ = 0$. With obvious notation,

$$\begin{aligned} a(u_1 - u_2, (u_1 - u_2)^+) &= \sum_{\tau \in T_h} \int_{\tau} \nabla(u_1 - u_2) \cdot \nabla(u_1 - u_2)^+ dx \\ &= \sum_{\tau \in T_h} E_{\tau}. \end{aligned}$$

Let A, B, C be the vertices of the triangle τ , and denote the corresponding nodal basis functions by ϕ_A, ϕ_B, ϕ_C . Then $(u_1 - u_2)|_{\tau} = v_A \phi_A + v_B \phi_B + v_C \phi_C$, where $v_A = u_1(A) - u_2(A)$, etc. We consider four different cases.

Case 1: $v_A \leq 0, v_B \leq 0, v_C \leq 0$. Here $(u_1 - u_2)^+|_{\tau} = 0$, which implies $E_{\tau} = 0$.

Case 2: $v_A > 0, v_B > 0, v_C > 0$. Clearly $E_{\tau} \geq 0$.

Case 3: $v_A \leq v_B \leq 0 < v_C$. This gives $(u_1 - u_2)^+|_{\tau} = v_C \phi_C$. So we have

$$\nabla(u_1 - u_2) \cdot \nabla(u_1 - u_2)^+|_{\tau} = v_A v_C \nabla \phi_A \cdot \nabla \phi_C + v_B v_C \nabla \phi_B \cdot \nabla \phi_C + v_C^2 |\nabla \phi_C|^2.$$

Since the angles of τ are at most $\frac{\pi}{2}$, we have $\nabla \phi_A \cdot \nabla \phi_C \leq 0$ and $\nabla \phi_B \cdot \nabla \phi_C \leq 0$.

Therefore

$$\nabla(u_1 - u_2) \cdot \nabla(u_1 - u_2)^+|_{\tau} > 0.$$

Hence, $E_{\tau} > 0$.

Case 4: $v_A \leq 0 < v_B \leq v_C$. In this case $(u_1 - u_2)^+|_{\tau} = v_B \phi_B + v_C \phi_C$. Using the identity $\nabla \phi_A + \nabla \phi_B + \nabla \phi_C = 0$, we see that

$$\begin{aligned} \nabla(u_1 - u_2)|_{\tau} &= (v_A - v_B) \nabla \phi_A + (v_C - v_B) \nabla \phi_C, \\ \nabla(u_1 - u_2)^+|_{\tau} &= -v_B \nabla \phi_A + (v_C - v_B) \nabla \phi_C. \end{aligned}$$

Computing the inner product we get

$$\begin{aligned} \nabla(u_1 - u_2) \cdot \nabla(u_1 - u_2)^+|_\tau &= -(v_A - v_B)v_B|\nabla\phi_A|^2 + (v_C - v_B)^2|\nabla\phi_C|^2 \\ &\quad + (v_A - v_B)(v_C - v_B)\nabla\phi_A \cdot \nabla\phi_C - v_B(v_C - v_B)\nabla\phi_A \cdot \nabla\phi_C \\ &> 0. \end{aligned}$$

Hence, $E_\tau > 0$.

For each $\tau \in T_h$ we can label the vertices such that one of the cases above holds. Since each E_τ is nonnegative, it then follows from (1.7) that

$$(u_1 - u_2)^+ = 0. \blacksquare$$

Remark: In general $K_1 \supseteq K_2$ does not imply the monotonicity property used above even if K_1, K_2 are induced by the obstacles Ψ_1, Ψ_2 , respectively, with $\Psi_1 \leq \Psi_2$. This is because *sup* and *inf* defined here are node-wise, not the usual continuous sense. However if Ψ_1 is piecewise linear with $\Psi_1 \leq \Psi_2$, then it is easy to see that $\{K_1, K_2\}$ does have the monotonicity property. The *sup* condition does not require anything beyond $\Psi_1 \leq \Psi_2$.

The following theorem is about the L^∞ -norm stability of the discrete solution of (1.2) for the perturbation of the piecewise linear obstacle function.

Theorem 1.2 *If K_1 and K_2 are two convex subsets of V_h which are generated by two piecewise linear obstacles Ψ_1 and Ψ_2 respectively, and u_1 and u_2 are the corresponding solutions of the variational inequalities (1.3) and (1.4), respectively, then*

$$|u_1 - u_2|_\infty \leq |\Psi_1 - \Psi_2|_\infty. \quad (1.8)$$

Proof. Let $r = |\Psi_1 - \Psi_2|_\infty$. Taking $v_h = u_1 + (u_2 - u_1 - r)^+ \in K_1$ in (1.3), and $v_h = u_2 - (u_2 - u_1 - r)^+ \in K_2$ in (1.4), by addition, we find

$$a(u_2 - u_1, (u_2 - u_1 - r)^+) \leq 0.$$

By using the idea in the proof of Theorem 1.1, we have

$$(u_2 - u_1 - r)^+ = 0$$

which implies $u_2 \leq u_1 + r$.

We also have $u_1 \leq u_2 + r$ by interchanging u_1 and u_2 . Hence (1.8). ■

Remark: If one of Ψ_1, Ψ_2 is not piecewise linear, the above theorem does not hold in general. Consider the case when $\Omega = (0, 1)$, $f = 0$, $\Psi_1 = 0$ and Ψ_2 has a peak near the boundary and equal to zero on the rest of the interval. If we take $h = \frac{1}{2}$, then $u_1 = 0$, but u_2 can be enormous.

We now briefly consider the five point finite difference approximation to the Laplacian. It is well known that this scheme is identical to the finite element approximation whose triangulation T_h is generated by dividing each rectangle of the finite difference mesh into two triangles. So the following similar monotonicity result is an easy corollary of Theorem 1.1.

Suppose m is the number of the interior mesh points. Let A^h be the $m \times m$ matrix generated by the five point approximation to $-\Delta$. Set

$$K^h = \{v^h \in R^m : v^h \geq \Psi^h\} \quad \text{for some } \Psi^h \in R^m. \quad (1.9)$$

The discrete obstacle problem is the following. Given $f^h \in R^m$, find $u^h \in K^h$ such that

$$(A^h u^h, v^h - u^h) \geq (f^h, v^h - u^h) \quad \text{for all } v^h \in K^h. \quad (1.10)$$

Theorem 1.3 *Suppose K_1^h and K_2^h are two convex sets which are induced by two vectors Ψ_1^h and Ψ_2^h as defined by (1.9), and u_1^h and u_2^h solve the corresponding variational inequalities like (1.10). If $\Psi_1^h \leq \Psi_2^h$ componentwise, then, in the same sense,*

$$u_1^h \leq u_2^h.$$

We now want to establish a relationship between the fully discrete and semidiscrete versions of the obstacle problems. We will use two additional variational inequalities. A new discrete obstacle $\tilde{\Psi}_h$ by shifting Ψ_h . Take

$$\tilde{\Psi}_h = \Psi_h + h^2 |\Psi|_{2,\infty}, \quad (1.11)$$

where $|\Psi|_{2,\infty} = \max\{\|\Psi_{11}\|_\infty, \|\Psi_{12}\|_\infty, \|\Psi_{22}\|_\infty\}$ and the subscripts here denote partial differentiation. We set $\tilde{K}_h = \{v_h \in V_h : v_h \geq \tilde{\Psi}_h\}$ and consider the following problem: Find $\tilde{u}_h \in \tilde{K}_h$, such that

$$a(\tilde{u}_h, v_h - \tilde{u}_h) \geq (f, v_h - \tilde{u}_h) \quad \text{for all } v_h \in \tilde{K}_h. \quad (1.12)$$

Note that since $\Psi < 0$ on Γ , \tilde{K} is not empty for h sufficiently small.

A semi-discrete obstacle problem is defined using $K^* = \{v_h \in V_h : v_h \geq \Psi\} = K \cap V_h$: Find $u^* \in K_h^*$, such that

$$a(u_h^*, v_h - u_h^*) \geq (f, v_h - u_h^*) \quad \text{for all } v_h \in K_h^*. \quad (1.13)$$

Notice that, in general, $K_h^* \neq K_h$. Also, K_h^* in generically involves infinitely many constraints; it is for this reason we call this is a semi-discrete problem.

To show u_h, u_h^* and \tilde{u}_h have a monotone relationship, we need to justify that both $\{K_h, K_h^*\}$ and $\{K_h^*, \tilde{K}_h\}$ satisfy the monotonicity property stated in Theorem 1.1. The following three lemmas are tools which are useful in making this justification. The first lemma is elementary.

Lemma 1.1 *Suppose that \mathbf{n} is a unit vector and that $\phi(t) = \Psi(P + t\mathbf{n})$ for some point P . Then*

$$|\phi''| \leq 2|\Psi|_{2,\infty}.$$

Lemma 1.2 *Suppose that $0 < \alpha \leq h$ and that $\phi \in W^{2,\infty}(0, \alpha)$. Suppose that μ is an affine function such that $\mu(0) \geq \phi(0) + \frac{1}{2}h^2\|\phi''\|_\infty$ and $\mu(\alpha) \geq \phi(\alpha)$. Then*

$$\mu \geq \phi \quad \text{on } (0, \alpha).$$

Proof. If the conclusion fails $\xi = \mu - \phi$ has a negative minimum at $t_0 \in (0, \alpha)$. By Taylor's theorem

$$\begin{aligned} \frac{1}{2}h^2\|\phi''\|_\infty &\leq \xi(0) \leq \xi(t_0) + \frac{1}{2}t_0^2\|\phi''\|_\infty \\ &< \frac{1}{2}t_0^2\|\phi''\|_\infty \leq \frac{1}{2}h^2\|\phi''\|_\infty. \quad \blacksquare \end{aligned}$$

Lemma 1.3 (a) For any $v_h \in K_h$ and $v_h^* \in K_h^*$, $\sup(v_h, v_h^*) \in K_h^*$ and $\inf(v_h, v_h^*) \in K_h$.

(b) For any $v_h^* \in K_h^*$ and $\tilde{v}_h \in \tilde{K}_h$, $\sup(v_h^*, \tilde{v}_h) \in \tilde{K}_h$ and $\inf(v_h^*, \tilde{v}_h) \in K_h^*$.

Proof. Part(a) is clear. We prove part(b).

Since $\sup(v_h^*, \tilde{v}_h) \geq \tilde{v}_h \geq \tilde{\Psi}_h$, we have $\sup(v_h^*, \tilde{v}_h) \in \tilde{K}_h$.

Now we show $\inf(v_h^*, \tilde{v}_h) \in K_h^*$, i.e., $\inf(v_h^*, \tilde{v}_h)$ does not cut through the original obstacle Ψ . For notational simplicity, let $w_h = \inf(v_h^*, \tilde{v}_h)$. Let A, B, C be the vertices of the triangle $\tau \in T_h$. Consider first the case in which $v_h^*(A) > \tilde{v}_h(A)$, $v_h^*(B) \leq \tilde{v}_h(B)$, $v_h^*(C) \leq \tilde{v}_h(C)$. For any point P on the edge (B, C) take $\mathbf{n} = (P - A)/|P - A|$. Define

$$\phi(t) = \Psi(A + t\mathbf{n})$$

and

$$\mu(t) = w_h(A + t\mathbf{n}).$$

By Lemma 1.1 and Lemma 1.2 $\mu \geq \phi$. Hence in this case $w_h \geq \Psi$. Next consider a triangle for which $v_h^* > \tilde{v}_h$ at more than one vertex. By the first case if shift the value down at one vertex we are still above Ψ . Hence we can replace v_h^* by the new function with the shifted value. We can now repeat this process once or twice more as required. ■

By part(b) of Lemma 1.3, one can observe that $\tilde{\Psi}_h$ stays above Ψ , which in effect gives a corollary, a theorem from classical approximation theory.

Corollary 1.1 If $w \in W^{2,\infty}(\Omega)$, then

$$|w - \pi_h w|_\infty \leq h^2 |w|_{2,\infty}.$$

By using Theorem 1.1 and Lemma 1.3, we have the following monotonicity theorem of the solutions of the discrete obstacle problems (1.2), (1.13) and (1.12).

Theorem 1.4 If u_h , u_h^* and \tilde{u}_h are the solutions of (1.2), (1.13) and (1.12) respectively, then

$$u_h \leq u_h^* \leq \tilde{u}_h.$$

1.3 L^∞ -Error Bound

In this section we apply the results of previous section to get a bound on the error in u_h in the max-norm. We say that a family of triangulations $\{T_h\}$ satisfies the shape regularity condition if there is a constant $\rho > 0$ independent of h , such that for any $\tau \in T_h$ there is a disk with radius ρh contained in τ . For the approximation u_h^* defined by (1.13), Nitsche [16] proved the following result.

Theorem 1.5 (Joachim Nitsche) *Suppose the family of triangulations satisfy both the maximum angle condition and the shape regularity condition. Assume Ψ and $u \in W^{2,\infty}(\Omega)$, then there exists a constant C independent of h , such that*

$$|u - u_h^*|_{0,\infty,\Omega} \leq C \cdot h^2 |\ln h| (\|u\|_{2,\infty,\Omega} + \|\Psi\|_{2,\infty,\Omega}).$$

This result together with Theorem 1.4 and Theorem 1.2 gives the following

Theorem 1.6 *Assume the conditions in Theorem 1.5 are satisfied. Then there exists a constant C independent of h , such that*

$$\begin{aligned} |u - u_h|_{0,\infty,\Omega} &\leq C \cdot h^2 |\ln h| (\|u\|_{2,\infty,\Omega} + \|\Psi\|_{2,\infty,\Omega}), \\ |u - \tilde{u}_h|_{0,\infty,\Omega} &\leq C \cdot h^2 |\ln h| (\|u\|_{2,\infty,\Omega} + \|\Psi\|_{2,\infty,\Omega}). \end{aligned}$$

Remark: In some situations, we may only interested in the approximations which stay above the original obstacle. Then the approximation \tilde{u}_h of (1.12) is a very good choice because it still stays close to the true solution u within an optimal error pointwise.

CHAPTER 2

**NUMERICAL APPROXIMATION OF
TIME-DEPENDENT FLOW OF BINGHAM FLUID
IN CYLINDRICAL PIPES**

2.1 Introduction

Numerical solution of time-dependent obstacle problems have been investigated by Berger and Falk [2], Donati [5], Fetter [8], Gastaldi and Gilardi [9], Jerome [13], Johnson [14], Scholz [17] and Vuik [18]. In this paper, we consider the numerical treatment of another type of parabolic variational inequality which describes the time-dependent flow of a Bingham fluid in a cylindrical pipe.

Let Ω be a bounded, convex domain in R^2 with C^2 boundary Γ . We define

$$\begin{aligned} V &= H_0^1(\Omega), \\ (v, w) &= \int_{\Omega} v w dx \quad \text{for } v, w \in L^2(\Omega), \\ a(v, w) &= \int_{\Omega} \nabla v \cdot \nabla w dx \quad \text{for } v, w \in V, \\ j(v) &= \int_{\Omega} |\nabla v| dx. \end{aligned}$$

Denote by $\|\cdot\|_s$ the norm on the Sobolev space $H^s(\Omega)$. It is well known that the form $a(v, w)$ is coercive on V , that is, there exists a constant $\alpha \in (0, 1)$ such that

$$a(v, v) \geq \alpha \|v\|_1^2 \quad \text{for } v \in V. \quad (2.1)$$

We shall study a method for finding approximate solutions of the following problem. Given functions f and u_0 and a nonnegative constant g , find $u : [0, T] \rightarrow V$ such that

$$\left(\frac{\partial u}{\partial t}, v - u\right) + a(u, v - u) + gj(v) - gj(u) \geq (f, v - u) \quad \text{for } v \in V, t \in (0, T] \quad (2.2)$$

$$u(x, 0) = u_0(x). \quad (2.3)$$

The function u needs to have some regularity for (2.2) to make sense. Here and in the corresponding finite-dimensional problem we consider later, we suppose that u is an $L^2(0, T)$ map into V and that $\frac{\partial u}{\partial t}$ is an $L^2(0, T)$ map into V' , the dual of V . It is shown in [7] that u exists in this class and that it has a natural trace at $t = 0$.

In Duvaut and Lions [7] it is shown that if u is the axial velocity in a laminar Bingham flow in a cylindrical duct of cross-section Ω , then u satisfies (2.2). In this context f is the axial pressure gradient, g is the stress threshold below which the fluid behaves as a rigid material and above which it behaves as an incompressible viscous fluid. In this paper we take the viscosity to be one. Many economically important substances are well approximated as Bingham fluids: heavy crude oils, drilling muds, cement slurries, and coal slurries. If g is strictly positive, rigid zones can exist in the interior of the flow. As g increases, these rigid zones become larger and may completely block the flow when g is sufficiently large.

In section 2 a numerical approximation to the nonregularized problem is defined and briefly examined; for this approximation the problem is discretized in space but not in time. In section 3 regularization is introduced and a collection of bounds for these functions are given. Some measures of the smoothness of solutions of the regularized problem are uniformly bounded as the regularization parameter goes to zero, and this gives regularity on the solution of (2.2). A backward differenced in time Galerkin in space scheme is given in section 4, and a priori error estimates are stated and proved. Section 5 is devoted to a closer study of these methods in the special case of a single space dimension; better theorems can be proved in this case. Some numerical results are given in section 6. In section 7 we present a modification of a beautiful result of H. Brezis; the result is used in section 3.

2.2 Continuous Time Finite Element Approximation

Let V_h be a finite-dimensional subspace of V . A continuous time finite element approximation is defined as follows:

Find $u_h : [0, T] \rightarrow V_h$ such that

$$\left(\frac{\partial u_h}{\partial t}, v_h - u_h\right) + a(u_h, v_h - u_h) + gj(v_h) - gj(u_h) \geq (f, v_h - u_h) \quad (2.4)$$

for $v_h \in V_h, 0 < t \leq T,$

$$u_h(0) = Q_h u_0, \quad (2.5)$$

where Q_h denotes the L^2 projection onto V_h .

Take the finite dimensional subspaces V_h to belong to a family \mathcal{V} such that certain properties hold. For each element of \mathcal{V} there is a positive number h (hence the notation V_h). Further the family \mathcal{V} is such that there exists a constant C such that for all $V_h \in \mathcal{V}$ and $v \in V \cap H^2(\Omega)$

$$\|v - Q_h v\|_k \leq Ch^{2-k} \|v\|_2, \quad k = 0, 1.$$

One can think of \mathcal{V} as consisting of spaces of continuous piecewise linear functions over triangles of size approximately h , where there is some regularity to the shape of the triangles. Note that showing that L^2 -projection approximates well in H^1 usually requires a, so-called, “inverse assumption”. In the remainder of this paper when we are dealing with finite dimensional spaces V_h and claim that a constant is independent of h , the constant may well depend on the family \mathcal{V} .

For functions $\phi : [0, T] \rightarrow X$ where X is a norm space with norm $\|\cdot\|_X$. We denote by $\|\phi\|_{L^p(X)}$ the $L^p[0, T]$ norm of $\|\phi(t)\|_X$. We also need, for $\delta \in [0, T]$, the $L^p[\delta, T]$ norm of $\|\phi(t)\|_X$ which we denote by $\|\phi\|_{L^p(\delta, T; X)}$. First we prove an a priori estimate for the solution of the semi-discrete problem (2.4).

Lemma 2.1 *Let u_h be the solution of (2.4). Then*

$$\|u_h\|_{L^2(H^1)} \leq \frac{1}{\alpha} (\|f\|_{L^2(H^{-1})} + \|u_0\|_0). \quad (2.6)$$

Proof. Taking $v_h = 0$ in (2.4), we get

$$\left(\frac{\partial u_h}{\partial t}, u_h\right) + a(u_h, u_h) + gj(u_h) \leq (f, u_h).$$

Noticing $j(u_h) \geq 0$ and that $g > 0$, we deduce that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_h\|_0^2 + \alpha \|u_h\|_1^2 &\leq \frac{1}{2} \frac{d}{dt} \|u_h\|_0^2 + a(u_h, u_h) \\ &\leq (f, u_h) \\ &\leq \|f\|_{-1} \|u_h\|_1 \\ &\leq \frac{1}{2\alpha} \|f\|_{-1}^2 + \frac{\alpha}{2} \|u_h\|_1^2. \end{aligned}$$

Integrating with respect to t and using $u_h(0) = Q_h u_0$ gives

$$\|u_h(T)\|_0^2 + \alpha \|u_h\|_{L^2(H^1)}^2 \leq \frac{1}{\alpha} \|f\|_{L^2(H^{-1})}^2 + \|Q_h u_0\|_0^2,$$

and this gives the conclusion since $\alpha < 1$. ■

The difference between the solution of u of (2.2) and u_h of (2.4) is estimated in the following theorem.

Theorem 2.1 *Let u and u_h , be defined by (2.2) and (2.4), respectively. Suppose that $u \in L^2(H^2)$ and $u_0 \in H_0^1(\Omega)$. Then*

$$\|u - u_h\|_{L^\infty(L^2)} + \|u - u_h\|_{L^2(H^1)} \leq C\sqrt{h}, \quad (2.7)$$

where C is independent of h .

Proof. Take $v_h = Q_h u$ in (2.4), $v = u_h$ in (2.2) and add them, to get

$$\begin{aligned} -\left(\frac{\partial}{\partial t}(u - u_h), u - u_h\right) + \left(\frac{\partial u_h}{\partial t}, Q_h u - u\right) - a(u - u_h, u - u_h) \\ + a(u_h, Q_h u - u) + gj(Q_h u) - gj(u) \geq (f, Q_h u - u). \end{aligned}$$

Since $(\frac{\partial u_h}{\partial t}, Q_h u - u) = 0$, we see that that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u - u_h\|_0^2 + \alpha \|u - u_h\|_1^2 &\leq a(u_h, Q_h u - u) \\ &\quad + gj(Q_h u) - gj(u) + (f, u - Q_h u) \\ &\leq \|u_h\|_1 \cdot \|Q_h u - u\|_1 \\ &\quad + g\sqrt{\text{measure}(\Omega)} \|Q_h u - u\|_1 \\ &\quad + \|f\|_0 \cdot \|u - Q_h u\|_0. \end{aligned}$$

Integration with respect to t yields

$$\begin{aligned} \|u - u_h\|_{L^\infty(L^2)}^2 + \|u - u_h\|_{L^2(H^1)}^2 &\leq C(\|u_0 - Q_h u_0\|_0^2 + \|u_h\|_{L^2(H^1)} \cdot \|Q_h u - u\|_{L^2(H^1)} \\ &\quad + g\sqrt{T \cdot \text{measure}(\Omega)} \|u - Q_h u\|_{L^2(H^1)} \\ &\quad + \|f\|_{L^2(L^2)} \cdot \|u - Q_h u\|_{L^2(L^2)}). \end{aligned}$$

The conclusion follows by Lemma 2.1 and the facts

$$\begin{aligned} \|Q_h u - u\|_{L^2(H^1)} &\leq Ch, \\ \|Q_h u - u\|_{L^2(L^2)} &\leq Ch^2, \\ \|Q_h u_0 - u_0\|_0^2 &\leq Ch^4. \blacksquare \end{aligned}$$

Remark: The proof of the above theorem required the $L^2(H^2)$ regularity assumption on the solution u of (2.2). In Theorem 2.10 of Section 3, we show that $u \in L^\infty(H^2)$ under certain conditions.

2.3 Method of Regularization

For $\tau \in R$ and $\epsilon > 0$ take

$$\phi_\epsilon(\tau) = \sqrt{\tau^2 + \epsilon^2};$$

then ϕ_ϵ is a C^∞ , convex, positive approximation to the absolute value function. We use

$$j_\epsilon(v) = \int_\Omega \phi_\epsilon(|\nabla v|) dx$$

as the regularization of the nondifferentiable term $\int_\Omega |\nabla v| dx$. For $\epsilon > 0$, the solution of the regularized problem is $u_\epsilon : [0, T] \rightarrow V$ such that

$$\left(\frac{\partial u_\epsilon}{\partial t}, v - u_\epsilon\right) + a(u_\epsilon, v - u_\epsilon) + g j_\epsilon(v) - g j_\epsilon(u_\epsilon) \geq (f, v - u_\epsilon) \quad \text{for } v \in V \text{ and } t \in (0, T] \quad (2.8)$$

$$u_\epsilon(x, 0) = u_0(x). \quad (2.9)$$

The regularization process defined above is justified by the following convergence result.

Theorem 2.2 *Let u_ϵ be the solution of (2.8) and u that of (2.2). Then*

$$\|u - u_\epsilon\|_{L^\infty(L^2)} + \|u - u_\epsilon\|_{L^2(H^1)} \leq C(\Omega, g, \alpha, T)\sqrt{\epsilon}, \quad (2.10)$$

where $C(\Omega, g, \alpha, T)$ is a constant only dependent of Ω, g, α, T .

Proof. From (2.2) and (2.8), we have

$$\begin{aligned} \left(\frac{\partial u}{\partial t}, u_\epsilon - u\right) + a(u, u_\epsilon - u) + gj(u_\epsilon) - gj(u) &\geq (f, u_\epsilon - u), \\ \left(\frac{\partial u_\epsilon}{\partial t}, u - u_\epsilon\right) + a(u_\epsilon, u - u_\epsilon) + gj_\epsilon(u) - gj_\epsilon(u_\epsilon) &\geq (f, u - u_\epsilon). \end{aligned}$$

By addition of the above two inequalities, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u - u_\epsilon\|_0^2 + \alpha \|u - u_\epsilon\|_1^2 &\leq gj_\epsilon(u) - gj(u) + gj(u_\epsilon) - gj_\epsilon(u_\epsilon) \\ &\leq g \cdot \text{measure}(\Omega)\epsilon, \end{aligned}$$

where we used the fact that

$$0 < \phi_\epsilon(\tau) - |\tau| \leq \epsilon.$$

Integrating with respect to t and using $u_\epsilon(x, 0) = u(x, 0) = u_0$ give (2.10). ■

Since $j_\epsilon(v)$ is differentiable, (2.8) has its equivalent equation form.

Theorem 2.3 *The solution u_ϵ for the regularized problem (2.8) is characterized by*

$$\left(\frac{\partial u_\epsilon}{\partial t}, v\right) + a(u_\epsilon, v) + g(\beta(u_\epsilon), v) = (f, v) \quad \text{for } v \in H_0^1(\Omega) \text{ and } t \in [0, T] \quad (2.11)$$

$$u_\epsilon(0, x) = u_0, \quad (2.12)$$

where

$$\begin{aligned} (\beta(u_\epsilon), v) &= (j'_\epsilon(u_\epsilon), v) \\ &= \int_\Omega \frac{\nabla u_\epsilon \cdot \nabla v}{\sqrt{\epsilon^2 + |\nabla u_\epsilon|^2}} dx. \end{aligned}$$

We can prove some a priori estimates for the solution u_ϵ of the regularized problem (2.8).

Theorem 2.4 *Suppose that $f \in L^2(L^2)$, $f_t \in L^2(H^{-1})$ and $u_0 \in H_0^1(\Omega)$. Let u_ϵ be the solution of the problem (2.8), the following estimates hold:*

$$\|u_\epsilon\|_{L^\infty(L^2)}^2 + \|u_\epsilon\|_{L^2(H^1)}^2 + \int_0^T (\beta(u_\epsilon), u_\epsilon) dt \leq C, \quad (2.13)$$

$$\|u'_\epsilon\|_{L^2(L^2)}^2 + \|u_\epsilon\|_{L^\infty(H^1)}^2 + \sup_{0 \leq t \leq T} \int_\Omega \sqrt{|\nabla u_\epsilon|^2 + \epsilon^2} dx \leq C, \quad (2.14)$$

$$\|u'_\epsilon\|_{L^\infty(\delta, T; L^2)}^2 + \|u'_\epsilon\|_{L^2(\delta, T; H^1)}^2 + \int_\delta^T \int_\Omega \frac{\epsilon^2 |\nabla u'_\epsilon|^2}{(\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{3}{2}}} dx dt \leq C(\delta), \quad (2.15)$$

where $u'_\epsilon = \frac{\partial u_\epsilon}{\partial t}$, δ is a positive constant and $C, C(\delta)$ are independent of ϵ .

Proof. By taking $v = u_\epsilon$ in (2.11) and integrating with respect to t , easily gives (2.13).

It is easy to verify that

$$(\beta(u_\epsilon), u'_\epsilon) = \frac{d}{dt} \int_\Omega \sqrt{|\nabla u_\epsilon|^2 + \epsilon^2} dx.$$

Taking $v = u'_\epsilon$ in (2.11), we get

$$\|u'_\epsilon\|_0^2 + \frac{1}{2} \frac{d}{dt} a(u_\epsilon, u_\epsilon) + g \frac{d}{dt} \int_\Omega \sqrt{|\nabla u_\epsilon|^2 + \epsilon^2} dx = (f, u'_\epsilon).$$

Again by integrating with respect to t and using (2.1), we get (2.14).

By differentiating (2.11) with respect to t , we get

$$(u''_\epsilon, v) + a(u'_\epsilon, v) + (\partial_t \beta(u_\epsilon), v) = (f_t, v). \quad (2.16)$$

Since

$$\begin{aligned} (\partial_t \beta(u_\epsilon), v) &= \partial_t \int_\Omega \frac{\nabla u_\epsilon \cdot \nabla v}{\sqrt{\epsilon^2 + |\nabla u_\epsilon|^2}} dx \\ &= \int_\Omega \frac{\nabla u'_\epsilon \cdot \nabla v}{\sqrt{\epsilon^2 + |\nabla u_\epsilon|^2}} dx - \int_\Omega \frac{(\nabla u_\epsilon \cdot \nabla v)(\nabla u_\epsilon \cdot \nabla u'_\epsilon)}{(\sqrt{\epsilon^2 + |\nabla u_\epsilon|^2})^3} dx, \end{aligned} \quad (2.17)$$

using Schwarz's inequality gives

$$(\partial_t \beta(u_\epsilon), u'_\epsilon) \geq \int_{\Omega} \frac{\epsilon^2 |\nabla u'_\epsilon|^2}{(\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{3}{2}}} dx.$$

Take $v = u'_\epsilon$ in (2.16) and use (2.1) to see that

$$\frac{1}{2} \frac{d}{dt} \|u'_\epsilon\|_0^2 + \alpha \|u'_\epsilon\|_1^2 + \int_{\Omega} \frac{\epsilon^2 |\nabla u'_\epsilon|^2}{(\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{3}{2}}} dx \leq (f_t, u'_\epsilon). \quad (2.18)$$

Integrating (2.18) from s to t gives

$$\|u'_\epsilon(t)\|_0^2 - \|u'_\epsilon(s)\|_0^2 + 2\alpha \int_s^t \|u'_\epsilon(\tau)\|_1^2 d\tau \leq 2 \int_s^t \|f_t(\tau)\|_{-1} \|u'_\epsilon(\tau)\|_1 d\tau,$$

from which we deduce that

$$\|u'_\epsilon(t)\|_0^2 - \|u'_\epsilon(s)\|_0^2 \leq \frac{1}{2\alpha} \int_s^t \|f_t(\tau)\|_{-1}^2 d\tau.$$

If we integrate the above inequality with respect to s from 0 to t and use (2.14), we have

$$\|u'_\epsilon(t)\|_0^2 \leq \frac{C}{t}.$$

In particular,

$$\|u'_\epsilon(\delta)\|_0^2 \leq \frac{C}{\delta}.$$

Integrating (2.18) from δ to t gives (2.15). ■

Theorem 2.5 *In addition to the conditions in Theorem 2.4, we assume that $u_0 \in H^2(\Omega)$ and satisfies*

$$\|\nabla \cdot \left(\frac{\nabla u_0}{\sqrt{\epsilon^2 + |\nabla u_0|^2}} \right)\|_0 \leq C, \quad (2.19)$$

with C independent of ϵ . Then

$$\|u'_\epsilon\|_{L^\infty(L^2)}^2 + \|u'_\epsilon\|_{L^2(H^1)}^2 + \int_0^T \int_{\Omega} \frac{\epsilon^2 |\nabla u'_\epsilon|^2}{(\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{3}{2}}} dx dt \leq C. \quad (2.20)$$

Proof. Taking $v = u'_\epsilon(0)$ in (2.11) with $t = 0$ and using (2.19), we get

$$\|u'_\epsilon(0)\|_0 \leq C,$$

with C independent of ϵ . Therefore we have (2.20) by integrating (2.18) from 0 to t .

■

Remark: For an important special case $u_0 = 0$, (2.19) is satisfied.

The following regularity theorem for u is similar to a theorem by Duvaut and Lions [7] (see pages 299-303); it follows from Theorem 2.4 and Theorem 2.5 by letting $\epsilon \rightarrow 0$.

Theorem 2.6 *We assume that f and u_0 satisfy the conditions of Theorem 2.5. Then the solution u of (2.2) satisfies*

$$\begin{aligned} u &\in L^\infty(H^1), \\ \frac{\partial u}{\partial t} &\in L^2(H^1) \cap L^\infty(L^2). \end{aligned}$$

In order to show the H^2 regularity of the solution u_ϵ of (2.8), we need a theorem regarding to H^2 regularity for the corresponding stationary problem which is similar to the following theorem of H. Brezis; for a proof see pages 118-122 of [3].

Theorem 2.7 (H.Brezis) *If $f \in L^2(\Omega)$, $w \in V$ satisfies*

$$a(w, v - w) + gj(v) - gj(w) \geq (f, v - w) \quad \text{for } v \in V,$$

then there exists a constant C , only dependent on the regularity of the operator $-\Delta$ on Ω , such that

$$\|w\|_2 \leq C\|f\|_0.$$

The proof of the next theorem closely parallels that of the previous one and is given in the appendix.

Theorem 2.8 *There exists a constant C , only dependent on the regularity of the operator $-\Delta$ on Ω , such that for $f \in L^2(\Omega)$ if $w_\epsilon \in V$ satisfies*

$$a(w_\epsilon, v - w_\epsilon) + gj_\epsilon(v) - gj_\epsilon(w_\epsilon) \geq (f, v - w_\epsilon) \quad \text{for } v \in V, \quad (2.21)$$

then

$$\|w_\epsilon\|_2 \leq C\|f\|_0.$$

Theorem 2.9 *If $f \in L^\infty(L^2)$ and the conditions of Theorem 2.5 are satisfied, then there exists a constant C independent of ϵ , such that if u_ϵ solves (2.8) then*

$$\|u_\epsilon\|_{L^\infty(H^2)} \leq C. \quad (2.22)$$

Proof. Since (2.20) gives a uniform bound on $\|u'_\epsilon\|_{L^\infty(L^2)}$, we can recast (2.8) with that term incorporated into f and apply Theorem 2.8. ■

From Theorems 2.6 and 2.7 we easily get

Theorem 2.10 *If $f \in L^\infty(L^2)$ and the conditions of Theorem 2.6 are satisfied, then*

$$u \in L^\infty(H^2).$$

2.4 Discretization of the regularized problem

In this section we combine the spatial discretization technique of section 2 (applied to the regularized problem) with a backward difference method to produce a fully discrete approximate solution process.

In deriving error estimates for the discrete solutions **in this section and the next we suppose hypotheses of Theorem 2.5, 2.6, 2.9 and 2.10 in the previous section hold for the solution u of (2.2) and the solution u_ϵ of (2.8).**

For N a positive integer, we set $\Delta t = T/N$, $t_k = k\Delta t$, $0 \leq k \leq N$. Then the fully discrete regularized problem is defined as follows: For $k = 1, \dots, N$, find $U^k \in V_h$, such that

$$(\partial U^k, v_h - U^{k+1}) + a(U^{k+1}, v_h - U^{k+1}) + gj_\epsilon(v_h) - gj_\epsilon(U^{k+1}) \geq (f^{k+1}, v_h - U^{k+1}) \quad (2.23)$$

for $v_h \in V_h$,

$$\|U^0 - u_0\|_0 \leq Ch, \quad (2.24)$$

where $\partial U^k = (U^{k+1} - U^k)/\Delta t$. Since we solve a minimization problem for a strictly convex functional at each time t_k , the solution exists and is unique.

The discrete norms used in this section are notationally cumbersome, so we define

$$\|\phi\|_{\Delta t} = \max_{0 \leq k \leq N} \|\phi^k\|_0 + (\sum_{k=1}^N \|\phi^k\|_1^2 \Delta t)^{\frac{1}{2}}.$$

We refer to this as the natural norm. For functions ϕ that are defined on $[0, T]$ we take $\phi^k = \phi(t_k)$. The main result of this section is the following.

Theorem 2.11 *Let u_ϵ and $\{U^0, U^1, \dots, U^N\}$ be the solutions of problem (2.8) and (2.23) respectively. Then there is a constant C independent of Δt , h , and ϵ , such that*

$$\|u_\epsilon - U\|_{\Delta t} \leq C(h^{\frac{1}{2}} + \frac{\Delta t}{\epsilon^{\frac{2}{3}}}).$$

Proof. Adopt the notation

$$e^k = u_\epsilon^k - U^k, \quad q^k = u_\epsilon^k - \tilde{u}^k,$$

with $\tilde{u} = Q_h u_\epsilon$ is the L^2 projection of u_ϵ to V_h . With $v_h = \tilde{u}^{k+1}$ in (2.23), we have

$$\begin{aligned} (-\partial U^k, q^{k+1} - e^{k+1}) + a(-U^{k+1}, q^{k+1} - e^{k+1}) + g j_\epsilon(\tilde{u}^{k+1}) - g j_\epsilon(U^{k+1}) \quad (2.25) \\ \geq (f^{k+1}, \tilde{u}^{k+1} - U^{k+1}). \end{aligned}$$

Taking $v = U^{k+1}$, $t = t_{k+1}$ in (2.8) gives

$$\begin{aligned} (\frac{\partial u_\epsilon}{\partial t}(t_{k+1}) - \partial u_\epsilon^k, -e^{k+1}) + (\partial u_\epsilon^k, -e^{k+1}) + a(u_\epsilon^{k+1}, -e^{k+1}) + g j_\epsilon(U^{k+1}) - g j_\epsilon(u_\epsilon^{k+1}) \quad (2.26) \\ \geq (f^{k+1}, U^{k+1} - u_\epsilon^{k+1}). \end{aligned}$$

Addition of (2.25) and (2.26) gives

$$\begin{aligned} (\partial e^k, e^{k+1}) + a(e^{k+1}, e^{k+1}) \leq & (-\partial U^k, q^{k+1}) + a(-U^{k+1}, q^{k+1}) + (f^{k+1}, q^{k+1}) \\ & + g j_\epsilon(\tilde{u}^{k+1}) - g j_\epsilon(u_\epsilon^{k+1}) \quad (2.27) \\ & + (\frac{\partial u_\epsilon}{\partial t}(t_{k+1}) - \partial u_\epsilon^k, -e^{k+1}). \end{aligned}$$

Notice that

$$\begin{aligned} (-\partial U^k, q^{k+1}) &= 0, \\ a(-U^{k+1}, q^{k+1}) &= a(e^{k+1}, q^{k+1}) - a(u_\epsilon^{k+1}, q^{k+1}), \\ gj_\epsilon(\tilde{u}^{k+1}) - gj_\epsilon(u_\epsilon^{k+1}) &\leq C\|q^{k+1}\|_1, \end{aligned}$$

and let

$$\rho^{k+1} = \frac{\partial u_\epsilon}{\partial t}(t_{k+1}) - \partial u_\epsilon^k.$$

The above relations give

$$\begin{aligned} (\partial e^k, e^{k+1}) + a(e^{k+1}, e^{k+1}) &\leq a(e^{k+1}, q^{k+1}) - a(u_\epsilon^{k+1}, q^{k+1}) + (f^{k+1}, q^{k+1}) \\ &\quad + C\|q^{k+1}\|_1 - (\rho^{k+1}, e^{k+1}) \\ &\leq \delta\|e^{k+1}\|_1^2 + \frac{1}{4\delta}\|q^{k+1}\|_1^2 + \|\Delta u_\epsilon^{k+1}\|_0\|q^{k+1}\|_0 \quad (2.28) \\ &\quad + \|f^{k+1}\|_0\|q^{k+1}\|_0 + C\|q^{k+1}\|_1 + |(\rho^{k+1}, e^{k+1})|. \end{aligned}$$

Choosing δ sufficiently small, multiplying by Δt in (2.28), and summing over k give

$$\begin{aligned} \sum_{k=0}^{n-1} (e^{k+1} - e^k, e^{k+1}) + \sum_{k=0}^{n-1} a(e^{k+1}, e^{k+1})\Delta t &\leq C\left(\sum_{k=0}^{n-1} \|q^{k+1}\|_1^2\Delta t \right. \\ &\quad \left. + \sum_{k=0}^{n-1} (\|\Delta u_\epsilon^{k+1}\|_0 + \|f^{k+1}\|_0)\|q^{k+1}\|_0\Delta t \right. \\ &\quad \left. + \sum_{k=0}^{n-1} \|q^{k+1}\|_1\Delta t + \sum_{k=0}^{n-1} |(\rho^{k+1}, e^{k+1})|\Delta t\right). \end{aligned}$$

Since

$$2\sum_{k=0}^n (e^{k+1} - e^k, e^{k+1}) = \|e^{n+1}\|_0^2 - \|e^0\|_0^2 + \sum_{k=0}^n \|e^{k+1} - e^k\|_0^2,$$

and by Theorem 2.9

$$\begin{aligned} \|q^{k+1}\|_1 &\leq Ch, \\ \|q^{k+1}\|_0 &\leq Ch^2, \end{aligned}$$

we have, by using (2.1) and Theorem 2.9,

$$\max_{0 \leq k \leq N} \|e^k\|_0^2 + \sum_{k=1}^N \|e^k\|_1^2 \Delta t \leq C(\|e^0\|_0^2 + h + \sum_{k=0}^{N-1} |(\rho^{k+1}, e^{k+1})| \Delta t). \quad (2.29)$$

Now let us estimate $\sum_{k=0}^{N-1} |(\rho^{k+1}, e^{k+1})| \Delta t$. Using (2.11) from Theorem 2.3 we get

$$\begin{aligned} (\rho^{k+1}, e^{k+1}) \Delta t &= (\partial u_\epsilon^k - \frac{\partial u_\epsilon}{\partial t}(t_{k+1}), e^{k+1}) \Delta t \\ &= (u_\epsilon(t_{k+1}) - u_\epsilon(t_k) - u'_\epsilon(t_{k+1}) \Delta t, e^{k+1}) \\ &= \int_{t_k}^{t_{k+1}} (u'_\epsilon(t) - u'_\epsilon(t_{k+1}), e^{k+1}) dt \\ &= \int_{t_k}^{t_{k+1}} a(u_\epsilon(t_{k+1}) - u_\epsilon(t), e^{k+1}) dt \\ &\quad + \int_{t_k}^{t_{k+1}} g(j'_\epsilon(u_\epsilon(t_{k+1})) - j'_\epsilon(u_\epsilon(t)), e^{k+1}) dt \\ &\quad - \int_{t_k}^{t_{k+1}} (f(t_{k+1}) - f(t), e^{k+1}) dt. \end{aligned}$$

We bound the first term on the right hand side by

$$\begin{aligned} \sum_{k=0}^{N-1} \left| \int_{t_k}^{t_{k+1}} a(u_\epsilon(t_{k+1}) - u_\epsilon(t), e^{k+1}) dt \right| &\leq \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} |a(u'_\epsilon(\tau), e^{k+1})| d\tau dt \\ &\leq \sum_{k=0}^{N-1} \|e^{k+1}\|_1 \Delta t \int_{t_k}^{t_{k+1}} \|u'_\epsilon(\tau)\|_1 d\tau \\ &\leq \sum_{k=0}^{N-1} (\delta \|e^{k+1}\|_1^2 \Delta t + \frac{1}{4\delta} (\Delta t)^2 \int_{t_k}^{t_{k+1}} \|u'_\epsilon(\tau)\|_1^2 d\tau) \\ &= \delta \sum_{k=0}^{N-1} \|e^{k+1}\|_1^2 \Delta t + \frac{1}{4\delta} (\Delta t)^2 \int_0^T \|u'_\epsilon(\tau)\|_1^2 d\tau. \end{aligned}$$

We can similarly estimate

$$\sum_{k=0}^{N-1} \left| \int_{t_k}^{t_{k+1}} (f(t_{k+1}) - f(t), e^{k+1}) dt \right| \leq \delta \sum_{k=0}^{N-1} \Delta t \|e^{k+1}\|_1^2 + \frac{1}{4\delta} (\Delta t)^2 \int_0^T \|f'(\tau)\|_{-1}^2 d\tau.$$

Now we estimate the middle term by

$$\left| \int_{t_k}^{t_{k+1}} (j'_\epsilon(u_\epsilon(t_{k+1})) - j'_\epsilon(u_\epsilon(t)), e^{k+1}) dt \right| \leq \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} |(\beta(u_\epsilon(\tau)))', e^{k+1}| d\tau dt.$$

From (2.17), we have

$$\begin{aligned}
(\beta(u_\epsilon(\tau))', e^{k+1}) &= \epsilon^{-\gamma} (\epsilon^\gamma \beta(u_\epsilon(\tau))', e^{k+1}) \\
&= \epsilon^{-\gamma} \int_{\Omega} \frac{\epsilon^\gamma \nabla u'_\epsilon \cdot \nabla e^{k+1}}{\sqrt{\epsilon^2 + |\nabla u_\epsilon|^2}} dx - \epsilon^{-\gamma} \int_{\Omega} \frac{\epsilon^\gamma (\nabla u_\epsilon \cdot \nabla e^{k+1}) (\nabla u_\epsilon \cdot \nabla u'_\epsilon)}{(\sqrt{\epsilon^2 + |\nabla u_\epsilon|^2})^3} dx, \\
&= \epsilon^{-\gamma} (I_1 + I_2).
\end{aligned}$$

with obvious notation and $0 \leq \gamma \leq 1$ to be chosen later. These terms satisfy

$$\begin{aligned}
|I_1| &\leq \left(\int_{\Omega} \frac{\epsilon^{2\gamma} |\nabla u'_\epsilon|^2}{\epsilon^2 + |\nabla u_\epsilon|^2} dx \right)^{\frac{1}{2}} \cdot \|e^{k+1}\|_1, \\
|I_2| &\leq \int_{\Omega} \frac{\epsilon^\gamma |\nabla e^{k+1}| \cdot |\nabla u'_\epsilon| \cdot |\nabla u_\epsilon|^2}{\sqrt{\epsilon^2 + |\nabla u_\epsilon|^2} (\epsilon^2 + |\nabla u_\epsilon|^2)} dx \\
&\leq \left(\int_{\Omega} \frac{\epsilon^{2\gamma} |\nabla u'_\epsilon|^2}{\epsilon^2 + |\nabla u_\epsilon|^2} dx \right)^{\frac{1}{2}} \cdot \|e^{k+1}\|_1.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} |(\beta(u_\epsilon(\tau))', e^{k+1})| d\tau dt &\leq 2\epsilon^{-\gamma} \Delta t \|e^{k+1}\|_1 \int_{t_k}^{t_{k+1}} \left(\int_{\Omega} \frac{\epsilon^{2\gamma} |\nabla u'_\epsilon|^2}{\epsilon^2 + |\nabla u_\epsilon|^2} dx \right)^{\frac{1}{2}} d\tau \\
&\leq \delta \Delta t \|e^{k+1}\|_1^2 + \frac{\Delta t^2}{\delta \epsilon^{2\gamma}} \int_{t_k}^{t_{k+1}} \int_{\Omega} \frac{\epsilon^{2\gamma} |\nabla u'_\epsilon|^2}{\epsilon^2 + |\nabla u_\epsilon|^2} dx d\tau.
\end{aligned}$$

We can conclude that

$$\begin{aligned}
&\sum_{k=0}^{N-1} \left| \int_{t_k}^{t_{k+1}} (j'_\epsilon(u_\epsilon(t_{k+1})) - j'_\epsilon(u_\epsilon(t)), e^{k+1}) dt \right| \\
&\leq \delta \sum_{k=0}^{N-1} \|e^{k+1}\|_1^2 \Delta t + \frac{\Delta t^2}{\delta \epsilon^{2\gamma}} \int_0^T \int_{\Omega} \frac{\epsilon^{2\gamma} |\nabla u'_\epsilon|^2}{\epsilon^2 + |\nabla u_\epsilon|^2} dx d\tau \\
&\leq \delta \sum_{k=0}^{N-1} \|e^{k+1}\|_1^2 \Delta t \\
&\quad + \frac{\Delta t^2}{\delta \epsilon^{2\gamma}} \left(\int_0^T \int_{\Omega} \frac{\epsilon^2 |\nabla u'_\epsilon|^2}{(\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{3}{2}}} dx d\tau \right)^{\frac{2}{3}} \left(\int_0^T \int_{\Omega} \epsilon^{6\gamma-4} |\nabla u'_\epsilon|^2 dx d\tau \right)^{\frac{1}{3}} \\
&\leq \delta \sum_{k=0}^{N-1} \|e^{k+1}\|_1^2 \Delta t + C \left(\frac{\Delta t}{\epsilon^{\frac{2}{3}}} \right)^2,
\end{aligned}$$

where in the last two steps we used the Holder inequality and then chose $\gamma = \frac{2}{3}$ and applied (2.20) of Theorem 2.5. Combining the above estimates gives

$$\begin{aligned} \sum_{k=0}^{N-1} |(\rho^{k+1}, e^{k+1})| \Delta t &\leq (2+g)\delta \sum_{k=0}^{N-1} \|e^{k+1}\|_1^2 \Delta t \\ &\quad + \frac{1}{4\delta} (\Delta t)^2 \|u'_\epsilon\|_{L^2(H^1)}^2 + C \left(\frac{\Delta t}{\epsilon^{\frac{2}{3}}}\right)^2 \\ &\quad + \frac{1}{4\delta} (\Delta t)^2 \|f'\|_{L^2(H^{-1})}^2 \end{aligned} \quad (2.30)$$

Since δ can be chosen sufficiently small, (2.24), (2.29), (2.30), and Theorem 2.5 complete the proof. ■

In order to estimate the error between the solution U of (2.23) and the solution u of (2.2), we need a modified version of Theorem 2.2.

Theorem 2.12 *Let u_ϵ be the solution of (2.8) and u that of (2.2). Then*

$$\|u - u_\epsilon\|_{\Delta t} \leq C(\epsilon^{\frac{1}{2}} + \epsilon^{\frac{1}{4}}(\Delta t)^{\frac{1}{2}}),$$

where C is independent of ϵ , h and Δt .

Proof. We set $z(t) = u(t) - u_\epsilon(t)$. Note that

$$\begin{aligned} \left| \sum_{k=1}^N \|z(t_k)\|_1^2 \Delta t - \int_0^T \|z(t)\|_1^2 dt \right| &= \left| \sum_{k=1}^N \int_{t_{k-1}}^{t_k} (\|z(t_k)\|_1^2 - \|z(t)\|_1^2) dt \right| \\ &= \left| \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \int_t^{t_k} \frac{d}{d\tau} \|z(\tau)\|_1^2 d\tau dt \right| \\ &\leq 2\Delta t \sum_{k=1}^N \int_{t_{k-1}}^{t_k} |a(z'(\tau), z(\tau)) + (z'(\tau), z(\tau))| d\tau \\ &\leq 2\Delta t \|z'\|_{L^2(H^1)} \|z\|_{L^2(H^1)} \\ &\leq 2\Delta t C(\Omega, g) \epsilon^{\frac{1}{2}} \|z'\|_{L^2(H^1)} \\ &\leq C\Delta t \epsilon^{\frac{1}{2}}, \end{aligned}$$

where in the last two steps we applied Theorem 2.2 and then Theorems 2.5 and 2.6. This result when combined with Theorem 2.2 completes the proof. ■

The conclusion that we draw from Theorems 2.11 and 2.12 is the following.

Theorem 2.13 *Let u be the the solution of (2.2), and $\{U^0, U^1, \dots, U^N\}$ be the solution of (2.23). Then there is a constant C independent of $\Delta t, h$ and ϵ , such that*

$$\|u - U\|_{\Delta t} \leq C(h^{\frac{1}{2}} + \frac{\Delta t}{\epsilon^{\frac{2}{3}}} + \epsilon^{\frac{1}{2}} + \epsilon^{\frac{1}{4}}(\Delta t)^{\frac{1}{2}}).$$

2.5 One-dimensional case

For the one-dimensional case using piecewise linear subspaces, better error estimates obtain.

First we need a lemma about interpolation and the functional j . For simplicity, let Ω be $(0, 1)$. Let $\{0 = x_0 < x_1 < \dots < x_M = 1\}$ partition $(0, 1)$ and take $h_i = x_i - x_{i-1}$.

Lemma 2.2 *If $r_h u$ is the linear nodal interpolant of $u \in H^1(0, 1)$, then*

$$j(r_h u) \leq j(u) \quad \text{and} \quad j_\epsilon(r_h u) \leq j_\epsilon(u)$$

Proof. We recall from section 3 that $\phi_\epsilon(\tau) = \sqrt{\tau^2 + \epsilon^2}$ and $j_\epsilon(u) = \int_0^1 \phi_\epsilon(\frac{du}{dx}) dx$. If we set $\phi(\tau) = |\tau|$, then $j(u) = \int_0^1 \phi(\frac{du}{dx}) dx$. Notice that ϕ and ϕ_ϵ are convex.

We will first prove $j_\epsilon(r_h u) \leq j_\epsilon(u)$. We show that $j_\epsilon(r_h u) \leq j_\epsilon(r_{\frac{h}{2}} u)$ with $r_{\frac{h}{2}} u$ is the $V_{\frac{h}{2}}$ -interpolant of u . The finer space is obtained from the coarser one by adding the midpoints $x_{i-\frac{1}{2}}$ to each interval (x_{i-1}, x_i) . Calculation shows

$$\begin{aligned} j_\epsilon(r_h u) &= \int_0^1 \phi_\epsilon(\frac{d}{dx} r_h u) dx \\ &= \sum_{i=1}^M \int_{x_{i-1}}^{x_i} \phi_\epsilon(\frac{d}{dx} r_h u) dx \\ &= \sum_{i=1}^M h_i \phi_\epsilon\left(\frac{u(x_i) - u(x_{i-1})}{h_i}\right) \\ &= \sum_{i=1}^M h_i \phi_\epsilon\left(\frac{1}{2}\left(\frac{u(x_{i-\frac{1}{2}}) - u(x_{i-1})}{\frac{1}{2}h_i} + \frac{u(x_i) - u(x_{i-\frac{1}{2}})}{\frac{1}{2}h_i}\right)\right) \\ &\leq \sum_{i=1}^M \left(\frac{1}{2} h_i \phi_\epsilon\left(\frac{u(x_{i-\frac{1}{2}}) - u(x_{i-1})}{\frac{1}{2}h_i}\right) + \frac{1}{2} h_i \phi_\epsilon\left(\frac{u(x_i) - u(x_{i-\frac{1}{2}})}{\frac{1}{2}h_i}\right)\right) \\ &= \sum_{i=1}^M \left(\int_{x_{i-1}}^{x_{i-\frac{1}{2}}} \phi_\epsilon\left(\frac{d}{dx} r_{\frac{h}{2}} u\right) dx + \int_{x_{i-\frac{1}{2}}}^{x_i} \phi_\epsilon\left(\frac{d}{dx} r_{\frac{h}{2}} u\right) dx\right) \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \phi_\epsilon \left(\frac{d}{dx} r_{\frac{h}{2}} u \right) dx \\
&= j_\epsilon \left(r_{\frac{h}{2}} u \right).
\end{aligned}$$

So we have

$$j_\epsilon(r_h u) \leq j_\epsilon \left(r_{\frac{h}{2^l}} u \right) \quad \text{for all natural numbers } l. \quad (2.31)$$

It is not hard to see that $r_h u$ converges to u in H_0^1 as $h \rightarrow 0$. (The derivative of the interpolant is the L^2 -projection of the derivative of u .) Hence $j_\epsilon(r_h u)$ also converges. Taking $l \rightarrow \infty$ in (2.31) gives $j_\epsilon(r_h u) \leq j_\epsilon(u)$. A similar proof gives $j(r_h u) \leq j(u)$.

■

The following theorem improves the semi-discrete error estimate given by Theorem 2.1 in this one-dimensional case.

Theorem 2.14 *Let u and u_h be as in Theorem 2.1, with Ω being an interval. Then*

$$\|u - u_h\|_{L^\infty(L^2)} + \|u - u_h\|_{L^2(H^1)} \leq Ch,$$

where C is independent of h .

Proof. We write $e_h = u - u_h$ and $\eta_h = u - r_h u$. By H^2 -regularity of u and u_0 , we have

$$\begin{aligned}
\|\eta_h\|_1 &\leq Ch \|u\|_2, \\
\|\eta_h\|_0 &\leq Ch^2 \|u\|_2, \\
\|e_h(0)\|_0 &= \|u_0 - Q_h u_0\|_0 \leq Ch^2 \|u_0\|_2,
\end{aligned}$$

with C independent of h . Further $r_h u$ is the elliptic projection of u in the sense that $a(\eta_h, v) = 0$ for any v in the space V . From the proof of Theorem 2.1 and Lemma 2.2, it is easy to see that

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|u - u_h\|_0^2 + \alpha \|u - u_h\|_1^2 &\leq (u'_h, r_h u - u) + (f, u - r_h u) + gj(r_h u) - gj(u) \\
&\leq (u'_h, r_h u - u) + (f, u - r_h u) \\
&= -(e'_h, \eta_h) + (u', \eta_h) + (f, \eta_h).
\end{aligned} \quad (2.32)$$

Integrating by parts gives

$$\begin{aligned} \int_0^t (e'_h, \eta_h) d\tau &= (e_h, \eta_h)|_0^t - \int_0^t (e_h, \eta'_h) ds \\ &\leq \frac{1}{4} \|e_h(t)\|_0^2 + \|\eta_h(t)\|_0^2 + \int_0^T |(e_h, \eta'_h)| ds + \|e_h(0)\|_0 \|\eta_h(0)\|_0. \end{aligned}$$

Since time derivative and r_h commute, we have $\|\eta'_h\|_0 \leq Ch\|u'\|_1$ and this gives

$$\begin{aligned} \int_0^T |(e_h, \eta'_h)| dt &\leq \int_0^T \|e_h\|_0 \cdot Ch\|u'\|_1 dt \\ &\leq \delta \|e_h\|_{L^2(L^2)}^2 + \frac{C^2 h^2}{4\delta} \|u'\|_{L^2(H^1)}^2. \end{aligned}$$

We can bound the last two terms in (2.32) as follows:

$$\begin{aligned} \left| \int_0^T (u', \eta_h) dt \right| &\leq Ch^2 \int_0^T \|u'\|_0 \|u\|_2 dt \\ &\leq Ch^2 \|u'\|_{L^2(L^2)} \|u\|_{L^2(H^2)}, \\ \left| \int_0^T (f, \eta_h) dt \right| &\leq \int_0^T \|f\|_0 Ch^2 \|u\|_2 dt \\ &\leq Ch^2 \|f\|_{L^2(L^2)} \|u\|_{L^2(L^2)}. \end{aligned}$$

By integrating with respect to t , choosing δ sufficiently small and using the above estimates, the conclusion of the theorem follows. ■

The final theorem of this note deals with the full discretization error estimate for the regularized problem in one-dimensional case.

Theorem 2.15 *Let u_ϵ and $\{U^0, U^1, \dots, U^N\}$ be as in Theorem 2.11 with Ω an interval. Then there is a constant C independent of Δt , h and ϵ , such that*

$$\|u_\epsilon - U\|_{\Delta t} \leq C \left(h + \frac{\Delta t}{\epsilon^{\frac{2}{3}}} \right).$$

Proof. We choose $\tilde{u} = r_h u_\epsilon$ is the nodal interpolant of u_ϵ . Set $q = u_\epsilon - \tilde{u}$, $q^k = u_\epsilon^k - \tilde{u}^k$, $e^k = u_\epsilon^k - U^k$, for $0 \leq k \leq N$, and use the notation of the proof of Theorem 2.11.

From (2.27) and Lemma 2.2, we get

$$\begin{aligned} (\partial e^k, e^{k+1}) + a(e^{k+1}, e^{k+1}) &\leq (-\partial U^k, q^{k+1}) + (f^{k+1}, q^k) \\ &\quad + \left(\frac{\partial u_\epsilon}{\partial t}(t_{k+1}) - \partial u_\epsilon^k, -e^{k+1} \right). \end{aligned}$$

We only need to estimate the first term since the estimate of the other terms is as in Theorem 2.11.

First notice that

$$\begin{aligned} (-\partial U^k, q^{k+1}) &= (\partial e^k, q^{k+1}) - (\partial u_\epsilon^k, q^{k+1}), \\ \sum_{k=0}^{N-1} |(\partial u_\epsilon^k, q^{k+1})| \Delta t &\leq \sum_{k=0}^{N-1} \|\partial u_\epsilon^k\|_0 \|q^{k+1}\|_0 \Delta t \\ &\leq Ch^2 \|u'_\epsilon\|_{L^\infty(L^2)} \|u_\epsilon\|_{L^\infty(H^2)}. \end{aligned}$$

Using summation by parts, we have

$$\begin{aligned} \sum_{k=0}^{n-1} (\partial e^k, q^{k+1}) \Delta t &= - \sum_{k=0}^{n-1} (e^k, \partial q^k) \Delta t + (e^n, q^n) - (e^0, q^0) \\ |(e^n, q^n)| &\leq \frac{1}{4} \|e^n\|_0^2 + \|q^n\|_0^2 \\ &\leq \frac{1}{4} \|e^n\|_0^2 + Ch^2 \|u_\epsilon\|_{L^\infty(H^1)}^2. \end{aligned}$$

Since ∂_t commutes with r_h , we have

$$\|q'\|_0 \leq Ch \|u'_\epsilon\|_1 \quad \text{for all } t \in [0, T].$$

Hence,

$$\begin{aligned} \|\partial q^k\|_0 &= \Delta t^{-1} \|q^{k+1} - q^k\|_0 \\ &\leq \Delta t^{-1} \int_{t_k}^{t_{k+1}} \left\| \frac{\partial q}{\partial t} \right\|_0 dt \\ &\leq C \Delta t^{-1} \int_{t_k}^{t_{k+1}} h \left\| \frac{\partial u_\epsilon}{\partial t} \right\|_1 dt \\ &\leq C \Delta t^{-\frac{1}{2}} h \left(\int_{t_k}^{t_{k+1}} \left\| \frac{\partial u_\epsilon}{\partial t} \right\|_1^2 dt \right)^{\frac{1}{2}}, \\ \sum_{k=0}^{N-1} |(e^k, \partial q^k)| \Delta t &\leq \sum_{k=0}^{N-1} \delta \|e^k\|_0^2 \Delta t + \sum_{k=0}^{N-1} \frac{1}{4\delta} \|\partial q^k\|_0^2 \Delta t \\ &\leq \delta \sum_{k=0}^{N-1} \|e^k\|_1^2 \Delta t + \frac{C}{4\delta} \sum_{k=0}^{N-1} h^2 \int_{t_k}^{t_{k+1}} \left\| \frac{\partial u_\epsilon}{\partial t} \right\|_1^2 dt \\ &= \delta \sum_{k=0}^{N-1} \|e^k\|_1^2 \Delta t + \frac{Ch^2}{4\delta} \left\| \frac{\partial u_\epsilon}{\partial t} \right\|_{L^2(H^1)}^2. \end{aligned}$$

Applying Theorem 2.5 and 2.9 to the above estimates and the rest estimates which are the same as Theorem 2.11, we can conclude the proof. ■

2.6 Numerical Experiment

We consider an one-dimensional example with $\Omega = (0, 1)$, $g = 1$, $f(t, x) = 10$, $u_0(x) = 0$, in (2.2). Let $\{0 = x_0 < x_1 < \dots, x_n = 1\}$ be a uniform partition on $(0, 1)$ with $x_i = ih$ for $0 \leq i \leq n$. Here h is the spatial mesh size. We take the time step size $\Delta t = h^2$. Take the finite-dimensional subspace V_h to be the continuous piecewise linear finite element space.

The numerical solution of (2.23) is denoted by $u_{h,\epsilon}$. Since the solution of (2.2) is not known we estimate the error with

$$E(h, \epsilon) = \|u_{h,\epsilon} - u_{h_0,\epsilon_0}\|_{\Delta t},$$

where $h_0 = 1/240$, $\epsilon_0 = 10^{-6}$.

We compute $E(h, \epsilon)$ for different values of h and ϵ . The results are gives in Table 1. We also list the corresponding $L^\infty(L^2)$ error in Table 2.

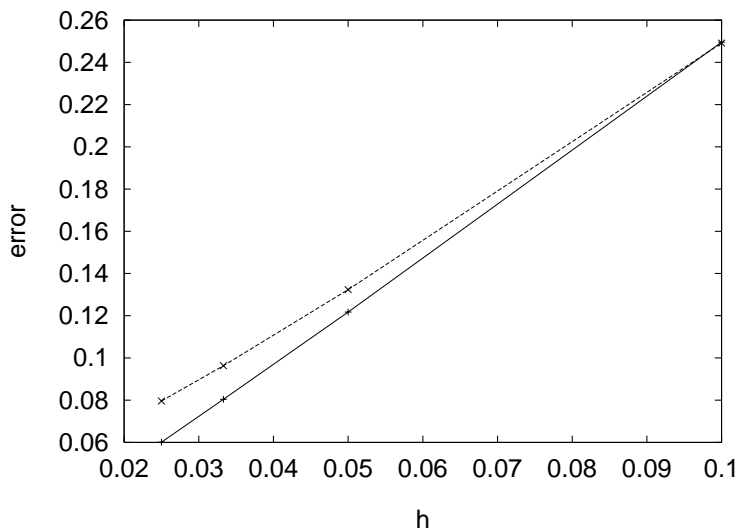
	$\epsilon = 1/10,000$	$\epsilon = 1/1,000$	$\epsilon = 1/100$	$\epsilon = 1/20$	$\epsilon = 1/10$	$\epsilon = 1/5$
$h = 1/40$	0.06050	0.06046	0.06012	0.06681	0.07958	0.10886
$h = 1/30$	0.08097	0.08093	0.08042	0.08565	0.09631	0.12181
$h = 1/20$	0.12217	0.12212	0.12171	0.12440	0.13236	0.15214
$h = 1/10$	0.24998	0.24993	0.24955	0.24846	0.24914	0.25895

Table 1: Natural Error Table

From these computations we see that when ϵ is small the error changes little with h fixed, which suggests our estimates may be conservative. We also see that when ϵ becomes bigger, the error is dominated by the size of ϵ .

In order to study how the error varies with h . We plot $E(h, \epsilon)$ versus h for fixed ϵ . Fig 1 shows that the total error is linearly proportional to h , which agrees

	$\epsilon = 1/10,000$	$\epsilon = 1/1,000$	$\epsilon = 1/100$	$\epsilon = 1/20$	$\epsilon = 1/10$	$\epsilon = 1/5$
$h = 1/40$	0.00067	0.00068	0.00061	0.00448	0.00879	0.01233
$h = 1/30$	0.00121	0.00118	0.00097	0.00433	0.00862	0.01608
$h = 1/20$	0.00252	0.00250	0.00233	0.00412	0.00824	0.01544
$h = 1/10$	0.01009	0.01012	0.00986	0.00874	0.00841	0.01367

Table 2: $L^\infty(L^2)$ Error TableFigure 1: The natural error curves. Solid line for $\epsilon = 0.01$, dashed line for $\epsilon = 0.1$.

with the theoretical estimate. We also plot the $L^\infty(L^2)$ error in Figs 2-3. From Figs 3 one can observe that the log error curve is approximately a straight line with slope equal to 2. This suggests that the $L^\infty(L^2)$ error is $O(h^2)$ when ϵ is small.

We plotted the solution curves for $t = 0.1, 0.2, \dots, 1$ with $h = 0.025$ in Fig 4. We see that there is a rigid zone in the center for all time and that when $t = 0.8$ the steady state has been reached.

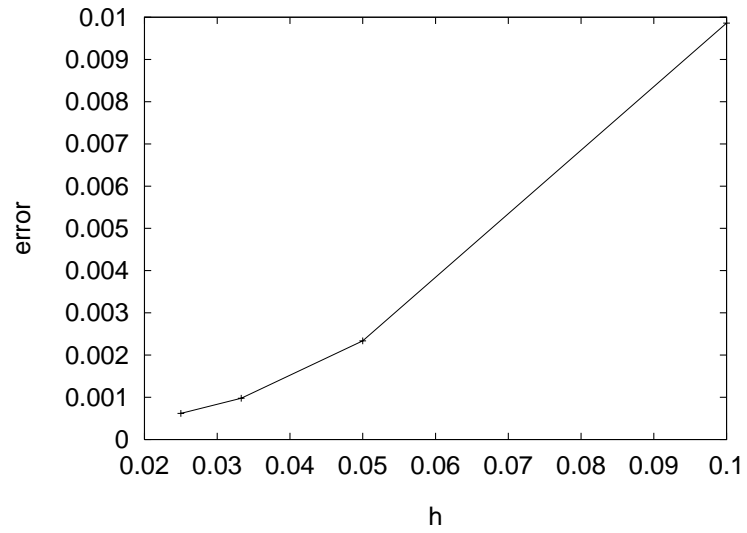


Figure 2: $L^\infty(L^2)$ error curve with $\epsilon = 0.01$.

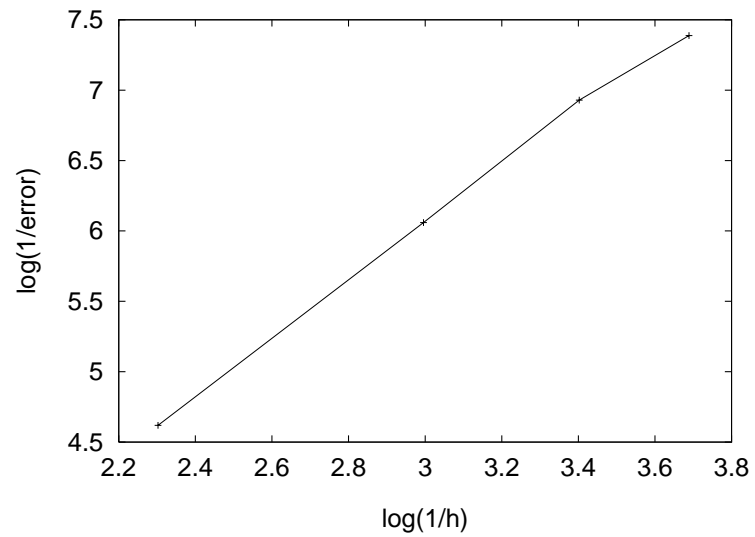


Figure 3: Log-log plot of $L^\infty(L^2)$ error with $\epsilon = 0.01$.

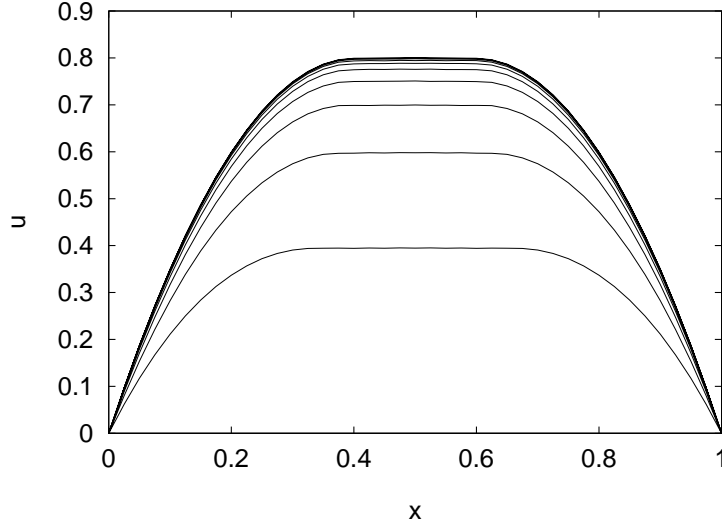


Figure 4: Solution Curves at times $0, 0.1, 0.2, \dots, 1.0$.

2.7 Proof of Theorem 2.8

In this section, we will prove Theorem 2.8 of section 3. The proof is adapted from the one by Brezis [3] with minor changes.

First we prove two lemmas.

Lemma 2.3 *Suppose $\Omega \in R^d$ is convex and that ϵ and δ are positive. Let $u \in H_0^1(\Omega)$ and let u_δ be the solution of*

$$\begin{aligned} u_\delta - \delta \Delta u_\delta &= u \quad \text{on } \Omega, \\ u_\delta &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Then we have

$$j_\epsilon(u_\delta) \leq j_\epsilon(u). \quad (2.33)$$

Proof. Let ζ be a smooth function such that $\zeta > 0$ on Ω , $\zeta = 0$ on Γ and $\frac{\partial \zeta}{\partial \nu} \neq 0$ on Γ . Here ν is the outward normal.

We first prove that, if v is a smooth function on $\bar{\Omega}$ which vanishes on Γ , then

$$\Delta v - \frac{\partial^2 v}{\partial \nu^2} = \frac{\partial v}{\partial \nu} \left(\frac{\partial \zeta}{\partial \nu} \right)^{-1} \left(\Delta \zeta - \frac{\partial^2 \zeta}{\partial \nu^2} \right) \quad \text{on } \Gamma. \quad (2.34)$$

We can assume that $0 \in \Gamma$ and we choose a coordinate system $(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_d)$ such that $\vec{e}_d = \vec{\nu}$. Locally, the equation of Γ is $x_d = f(x')$. In a neighborhood of 0 in R^{d-1} we have $\zeta(x', f(x')) \equiv 0$ and $v(x', f(x')) \equiv 0$. Thus, for $1 \leq i \leq d-1$,

$$\begin{aligned} \frac{\partial \zeta}{\partial x_i}(x', f(x')) + \frac{\partial \zeta}{\partial x_d}(x', f(x')) \frac{\partial f}{\partial x_i}(x') &\equiv 0, \\ \frac{\partial^2 \zeta}{\partial x_i^2}(x', f(x')) + 2 \frac{\partial^2 \zeta}{\partial x_d \partial x_i}(x', f(x')) \frac{\partial f}{\partial x_i}(x') + \\ \frac{\partial^2 \zeta}{\partial x_d^2}(x', f(x')) \left(\frac{\partial f}{\partial x_i}(x') \right)^2 + \frac{\partial \zeta}{\partial x_d}(x', f(x')) \frac{\partial^2 f}{\partial x_i^2}(x') &\equiv 0, \end{aligned}$$

Since $\frac{\partial f}{\partial x_i}(0) = 0$, we have

$$\frac{\partial^2 \zeta}{\partial x_i^2}(0, 0) + \frac{\partial \zeta}{\partial \nu}(0, 0) \frac{\partial^2 f}{\partial x_i^2}(0) = 0,$$

and hence

$$\Delta \zeta(0, 0) - \frac{\partial^2 \zeta}{\partial \nu^2}(0, 0) + \frac{\partial \zeta}{\partial \nu}(0, 0) \sum_{i=1}^{d-1} \frac{\partial^2 f}{\partial x_i^2}(0) = 0.$$

Similarly

$$\Delta v(0, 0) - \frac{\partial^2 v}{\partial \nu^2}(0, 0) + \frac{\partial v}{\partial \nu}(0, 0) \sum_{i=1}^{d-1} \frac{\partial^2 f}{\partial x_i^2}(0) = 0.$$

These two relations clearly give (2.34) when the sum involving f is eliminated. Since Ω is convex, ζ can be chosen to be concave so that

$$\Delta \zeta - \frac{\partial^2 \zeta}{\partial \nu^2} \leq 0 \text{ and } \frac{\partial \zeta}{\partial \nu} < 0. \quad (2.35)$$

By continuity it is sufficient to establish (2.33) for smooth u . Define for $\xi \in R^d$, the function $\psi_\epsilon(\xi) = \sqrt{|\xi|^2 + \epsilon^2}$. Since ψ_ϵ is convex, we have

$$\begin{aligned} \psi_\epsilon(\nabla u) - \psi_\epsilon(\nabla u_\delta) &\geq \sum_{k=1}^d \frac{\partial \psi_\epsilon}{\partial \xi_k}(\nabla u_\delta) \left(\frac{\partial u}{\partial x_k} - \frac{\partial u_\delta}{\partial x_k} \right) \\ &= \sum_{k=1}^d \frac{\partial \psi_\epsilon}{\partial \xi_k}(\nabla u_\delta) \left(-\delta \Delta \frac{\partial u_\delta}{\partial x_k} \right). \end{aligned}$$

Hence, using integration by parts,

$$\begin{aligned}
j_\epsilon(u_\delta) &\leq j_\epsilon(u) - \delta \sum_{k,l,m} \int_{\Omega} \frac{\partial^2 \psi_\epsilon}{\partial \xi_m \partial \xi_k} (\nabla u_\delta) \frac{\partial^2 u_\delta}{\partial x_l \partial x_m} \frac{\partial^2 u_\delta}{\partial x_l \partial x_k} dx \\
&\quad + \sum_{k=1}^d \int_{\Gamma} \frac{\partial \psi_\epsilon}{\partial \xi_k} (\nabla u_\delta) \frac{\partial^2 u_\delta}{\partial x_k \partial \nu} d\Gamma \\
&\leq j_\epsilon(u) + \delta \int_{\Gamma} \frac{1}{\sqrt{|\nabla u_\delta|^2 + \epsilon^2}} \frac{\partial u_\delta}{\partial \nu} \frac{\partial^2 u_\delta}{\partial \nu^2} d\Gamma
\end{aligned}$$

by noticing the convexity of ψ_ϵ and $u_\delta = 0$ on Γ .

Since $\Delta u_\delta = 0$ on Γ , (2.35) and (2.34), we have $\frac{\partial u_\delta}{\partial \nu} \frac{\partial^2 u_\delta}{\partial \nu^2} \leq 0$ on Γ so that

$$j_\epsilon(u_\delta) \leq j_\epsilon(u). \quad \blacksquare$$

Lemma 2.4 *Relation (2.21) in Theorem 2.8 is equivalent to the following:*

$$a(v, v - w_\epsilon) + gj_\epsilon(v) - gj_\epsilon(w_\epsilon) \geq (f, v - w_\epsilon) \quad \text{for } v \in V. \quad (2.36)$$

Proof. If (2.21) is satisfied then

$$\begin{aligned}
&a(v, v - w_\epsilon) + gj_\epsilon(v) - gj_\epsilon(w_\epsilon) - (f, v - w_\epsilon) \\
&= a(w_\epsilon, v - w_\epsilon) + gj_\epsilon(v) - gj_\epsilon(w_\epsilon) - (f, v - w_\epsilon) + a(v - w_\epsilon, v - w_\epsilon) \geq 0,
\end{aligned}$$

and hence (2.36) holds.

Conversely, letting $v = w_\epsilon + \theta(z - w_\epsilon)$ in (2.36), dividing by θ , and using the convexity of j_ϵ we see that

$$a(w_\epsilon + \theta(z - w_\epsilon), z - w_\epsilon) + gj_\epsilon(z) - gj_\epsilon(w_\epsilon) \geq (f, z - w_\epsilon).$$

By making $\theta \rightarrow 0$ we deduce (2.21). \blacksquare

Proof of Theorem 2.8:

Taking $v = u_\delta$ in (2.36), as in Lemma 2.3 with $u = w_\epsilon$, we obtain by using Lemma 2.4 and Lemma 2.3

$$\int_{\Omega} -\Delta u_\delta \delta \Delta u_\delta dx + gj_\epsilon(u_\delta) - gj_\epsilon(w_\epsilon) \geq (f, \delta \Delta u_\delta),$$

and

$$\int_{\Omega} |\Delta u_{\delta}|^2 dx \leq \int_{\Omega} |f| |\Delta u_{\delta}| dx.$$

So we have

$$\|\Delta u_{\delta}\|_0 \leq \|f\|_0 \text{ and } \|u_{\delta}\|_2 \leq C \|f\|_0$$

where C only depends on the regularity of $-\Delta$ on Ω .

Consequently, with the same C as above,

$$\|w_{\epsilon}\|_2 \leq C \|f\|_0. \blacksquare$$

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