A posterior Error Analysis of a Two-level Scheme for Solving the Obstacle Problem

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Abstract

A two-level algorithm is established for a discrete obstacle problem which is defined by a piecewise linear finite element discretization of a continuous problem. An $H^1$-convergence theorem of this method is also proved, in which the error estimate is independent of the true discrete solution.

Keywords: Obstacle problem, Two-level scheme, Error analysis, Finite element, $H^2$- regularity

1 Introduction

Let $\Omega$ be a convex bounded open subset of $\mathbb{R}^2$ with boundary $\partial \Omega$. Set $V = H^1_0(\Omega)$. We define $a(v, w)$ to be the energy bilinear form by

$$a(v, w) = \int_{\Omega} \nabla v \cdot \nabla w \, dx, \quad v, w \in V.$$ 

The form $a(v, w)$ is continuous on $V$ and

$$|a(v, w)| \leq \|v\|_1 \|w\|_1, \quad v, w \in V,$$

where $\| \cdot \|_1$ denote the $H^1$-norm.

It is well known that the form $a(v, w)$ is coercive on $V$, that is, there exists a constant $\gamma > 0$, such that

$$a(v, v) \geq \gamma \|v\|_1^2 \quad \text{for} \quad v \in V. \quad (1)$$
Let $\Psi \in H^1_0(\Omega) \cap H^2(\Omega)$ be the obstacle. We set $K = \{ v \in V : v \geq \Psi \quad \text{in} \quad \Omega \}$. Given $f \in L^2(\Omega)$, we define the energy

$$J(v) = \frac{1}{2} a(v,v) - (f,v),$$

where $(\cdot, \cdot)$ is the usual $L^2$-inner product. The obstacle problem consists of finding $u \in K$ such that

$$J(u) = \inf_{v \in K} J(v).$$

(2)

It is well known that the minimization problem (2) is equivalent to the following variational inequality. Find $u \in K$ such that

$$a(u, v - u) \geq (f, v - u) \quad \text{for} \quad v \in K.$$  

(3)

Obstacle problems are a type of free boundary problem. They are of interest both for their intrinsic beauty and for the wide range of applications they describe in subjects from physics to finance. Many important problems can be formulated by transformation to an obstacle problem, e.g., the filtration dam problem [6], the Stefan problem [6], the subsonic flow problem [4], American options pricing model [8], etc. The basic properties of the solution, including existence and uniqueness, were established by Lions and Stampacchia [5].

Since obstacle problems are highly nonlinear, the computation of approximate solutions can be difficult and expensive. Multilevel (or multigrid) methods have been proved very robust in modern scientific computing since the proof of convergence of the multigrid method in linear equations by Bank and Dupont [1]. The purpose of this paper is to develop a two level method to solve the obstacle problem and to establish the convergence theorems of this method.

## 2 $H^2$-regularity of The Obstacle Problem

The $H^2$-regularity of the solution $u$ of (3) in the case when the obstacle is flat, i.e. $\Psi = 0$, was discussed by Scholz [7]. The following lemma is a trivial corollary of Lemma 3 in [7].

**Lemma 1** Assume $f \in L^2(\Omega)$, $\Psi = 0$. Let $u$ be the solution of (3). Then there exists a constant $C$ only depending on the regularity of $\Omega$ such that

$$\|u\|_2 \leq C \|f\|_0.$$  

For general $\Psi$, we have the following $H^2$-regularity theorem.
**Theorem 2** Assume $f \in L^2(\Omega)$ and $\Psi \in H_0^1(\Omega) \cap H^2(\Omega)$. Then there exists a constant $C$ only depending on the regularity of $\Omega$ such that

$$
\|u\|_2 \leq C(\|f\|_0 + \|\Psi\|_2),
$$

$$
|u|_2 \leq C(\|f\|_0 + |\Psi|_2),
$$

where $|\cdot|_2$ is the $H^2$-seminorm.

**Proof.** From (3), we have

$$
\alpha(u - \Psi, v - \Psi - (u - \Psi)) \geq (f, v - \Psi - (u - \Psi)) - \alpha(\Psi, v - \Psi - (u - \Psi)) = (f + \Delta \Psi, v - \Psi - (u - \Psi)).
$$

Set $K_0 = \{v \in V : v \geq 0\}$. Take $u_1 = u - \Psi$ and $v_1 = v - \Psi$. Then $u_1 \in K_0$ satisfies

$$
\alpha(u_1, v_1 - u_1) \geq (f + \Delta \Psi, v_1 - u_1) \text{ for all } v_1 \in K_0.
$$

From Lemma 1, we have

$$
\|u_1\|_2 \leq C\|f + \Delta \Psi\|_0 \leq C(\|f\|_0 + |\Psi|_2).
$$

Therefore

$$
\|u\|_2 \leq \|u_1\|_2 + \|\Psi\|_2 \leq C(\|f\|_0 + \|\Psi\|_2),
$$

$$
|u|_2 \leq |u_1|_2 + |\Psi|_2 \leq C(\|f\|_0 + |\Psi|_2).
$$

**3 Two-level scheme**

We consider a sequence of triangulations $\{\mathcal{T}_k\}$ of the polygonal domain $\Omega$ determined as follows. Suppose $T_1$ is given and let $T_k$, $k \geq 2$, be obtained from $T_{k-1}$ via a systematic subdivision: edge midpoints in $T_{k-1}$ are connected by new edges to form $T_k$. Assume that $\{T_k\}$ satisfy the shape regularity condition which is stated as the following: there is a constant $\rho$ such that for any $\tau \in \mathcal{T}_k$ there is a disk $B$ of radius $\rho h$ with $B \subseteq \tau$. Let $V_k$ denote the finite element space of continuous piecewise linear functions with respect to $\mathcal{T}_k$ that vanish on $\partial \Omega$. Note that

$$
T_k \supset T_{k-1} \Rightarrow V_{k-1} \subset V_k.
$$
Let $h_k$ be the mesh size of $T_k$, then we have
\[ h_k = 2^{-(k-1)}h_1, \quad k = 1, 2, \ldots \]
For $h = h_k$, set $V_h = V_k$ and $V_{2h} = V_{k-1}$. We define the discrete obstacle function
\[ \Psi_h = \pi_h(\Psi), \]
where $\pi_h$ is the nodal interpolation operator: $H^1(\Omega) \to V_h$.
Set $K_h = \{v_h \in V_h : v_h(x) \geq \Psi_h(x), x \in \Omega\}$. The discrete obstacle problem is defined by finding $u_h \in K_h$ such that
\[ J(u_h) = \inf_{v_h \in K_h} J_h(v_h), \quad (4) \]
which is also characterized by
\[ a(u_h, v_h - u_h) \geq (f, v_h - u_h) \quad \text{for all } v_h \in K_h. \quad (5) \]
First we introduce a symmetric and positive definite operator $A_h : V_h \to V_h$ by the relation
\[ (A_h v_h, w_h) = a(v_h, w_h) \quad \text{for all } v_h, w_h \in V_h. \]
Suppose $\tilde{u}_h^1$ is the initial approximation to the solution $u_h$ of (4) after some smoothing process. Let
\[ e_h = u_h - \tilde{u}_h^1 \]
be the initial error and
\[ r_h = f - A_h \tilde{u}_h^1 \]
be the residual function. Set $\bar{K}_h = \{v_h \in V_h : v_h(x) \geq \Psi_h(x) - \tilde{u}_h^1(x), x \in \Omega\}$.
We define a new energy
\[ \bar{J}(v) = \frac{1}{2}a(v, v) - (r_h, v) \quad \text{for all } v \in V. \]

**Lemma 3** The error $e_h$ is in $\bar{K}_h$ such that
\[ \bar{J}(e_h) = \inf_{\phi_h \in \bar{K}_h} \bar{J}(\phi_h). \quad (6) \]

**Proof.** From (4), we have
\[ J(u_h^1 + e_h) = \inf_{\phi_h \in \bar{K}_h} J(u_h^1 + \phi_h). \]
The conclusion follows from

\[ J(\tilde{u}_h + \phi_h) = \frac{1}{2} \alpha(\tilde{u}_h^1 + \phi_h, \tilde{u}_h^1 + \phi_h) - (f, \tilde{u}_h^1 + \phi_h) \]
\[ \quad = \frac{1}{2} J(\tilde{u}_h^1) + \frac{1}{2} \alpha(\phi_h, \phi_h) - [(f, \phi_h) - \alpha(\tilde{u}_h^1, \phi_h)] \]
\[ \quad = \frac{1}{2} J(\tilde{u}_h^1) + \bar{J}(\phi_h). \]

Let \( \tilde{K}_{2h} \) be the nodal interpolation of \( \tilde{K}_h \) at the \( 2h \) grid. Define \( e_{2h} \in \tilde{K}_{2h} \) such that

\[ \bar{J}(e_{2h}) = \inf_{\phi_{2h} \in \tilde{K}_{2h}} \bar{J}(\phi_{2h}) \]  

(7)

We define the two-level scheme as the following.

Step 1. (Presmoothing step) Starting \( \tilde{u}_h^0 \), let

\[ \tilde{u}_h^1 = (G_h)^{m_1}(\tilde{u}_h^0). \]

Step 2. (Correction step)

\[ \tilde{u}_h^2 = \tilde{u}_h^1 + e_{2h}. \]  

(8)

Step 3. (Postsmoothing step)

\[ \tilde{u}_h^3 = (G_h)^{m_2}(\tilde{u}_h^2). \]

Here \( m_1 \) and \( m_2 \) are two natural numbers. \( G_h \) is a relaxation process, e.g., the one in [9].

Now we perform the error analysis for the correction step. The following result is modified from a result by Falk [3] which also can be found in [2]. We give a more detailed information on how \( C(u, f, \Psi) \) in his result depends on the parameters. More specifically, we give a formula for \( C(f, \Psi) \) which is independent of \( u \) because of the regularity assumption on \( \Omega \). The following proof is different from the original version.

Theorem 4 Assume \( f \in L^2(\Omega), \Psi \in H^1_0(\Omega) \cap H^2(\Omega) \). Let \( u \) and \( u_h \) be the solutions for the continuous and discrete obstacle problems (3), (5), respectively. Then

\[ \|u - u_h\|_1 \leq C(\|f\|_0 + |\Psi|_2)h \]  

(9)

where \( C \) is a constant only depending on \( \gamma \) and the regularity of \( \Omega \).
Proof. Taking \(v = u_h^* = \max\{u_h, \Psi\} \in K\) in (3) and \(v_h = \pi_h u \in K_h\) in (5), we have

\[
(f, u_h^* - u) \leq a(u, u_h^* - u) = a(u, u_h - u) + a(u, u_h^* - u_h), \quad (10)
\]

\[
(f, \pi_h u - u_h) \leq a(u_h, \pi_h u - u_h) = a(u_h, u - u_h) + a(u_h, \pi_h u - u). \quad (11)
\]

Addition of (10) and (11) gives

\[
(f, u_h^* - u_h) + (f, \pi_h u - u) \leq -a(u - u_h, \pi_h u - u_h) + a(u, u_h^* - u_h) + a(u, \pi_h u - u) + a(u_h - u, \pi_h u - u).
\]

By (1) and \(H^2\)-regularity of \(u\), we have

\[
\gamma \|u - u_h\|_1^2 \leq (-\Delta u - f, u_h^* - u_h) + (-\Delta u - f, \pi_h u - u)
\]

\[
+ \|u_h - u\|_1 \cdot \|\pi_h u - u\|_1
\]

\[
\leq \| - \Delta u - f \|_0 \cdot (\|u_h^* - u_h\|_0 + \|\pi_h u - u\|_0)
\]

\[
+ \frac{\gamma}{2} \|u_h - u\|_1^2 + \frac{1}{2\gamma} \|\pi_h u - u\|_1^2.
\]

Since

\[
\|u_h - u_h\|_0 = \|u_h - u_h\|_0 \{u_h < \psi\}
\]

\[
= \|u_h - \Psi\|_0 \{u_h < \psi\}
\]

\[
\leq \|\Psi_h - \Psi\|_0 \{u_h < \psi\}
\]

\[
\leq \|\Psi_h - \Psi\|_0
\]

\[
= \|\Psi - \pi_h \Psi\|_0
\]

\[
\leq C h^2 |\Psi|_2,
\]

and

\[
\|\pi_h - u\|_0 \leq C h^2 |u|_2,
\]

\[
\|\pi_h - u\|_1^2 \leq C h^2 |u|_2^2,
\]

we deduce that

\[
\|u - u_h\|_1^2 \leq C h^2 (|u|_2^2 + f\|_0 + \|\Psi\|_0^2).
\]

Using Theorem 2 gives

\[
\|u - u_h\|_1 \leq C h (\|f\|_0 + |\Psi|_2),
\]
with $C$ only depending on $\gamma$ and regularity of $\Omega$. □

Before we analyze the error after the correction step, we need an additional assumption on the family of triangulations $T_h$. We introduce a discrete $H^2$-seminorm

$$\|v_h\|_2 = (A_h^2 v_h, v_h)^{\frac{1}{2}}.$$

We assume the sequence of triangulations $\{T_h\}$ satisfy the following property: there is a constant $\beta$ only depending on the shape of the mesh such that for any $v_h \in V_h$ there exists a $v \in H^2(\Omega)$ such that $\pi_h v = v_h$ and $|v|_2 \leq \beta |v_h|_2$. We call a family of triangulations is regular if both the above property and the shape regularity condition defined in Section 3 are satisfied.

For the purpose of computation we like to express the discrete $H^2$-norm in term of the inner product in Euclidean space. Let $p_i, i = 1, \ldots, s$ be the vertices of $T_h$ that are in $\Omega$; these are usually called the interior vertices. Take $\phi_i \in V_h$ to be such that $\phi_i$ vanishes at all vertices of $T_h$ except $p_i$ and $\phi_i(p_i) = 1$. Then $V_h = \text{span}\{\phi_i\}$, and the $\phi_i$’s provide the usual nodal basis for $V_h$.

Define the $s \times s$ matrix $A_h = (a_{ij}^h)$ and the $s \times s$ matrix $B_h = (b_{ij}^h)$ by

$$a_{ij}^h = \alpha(\phi_j, \phi_i),$$
$$b_{ij}^h = \beta(\phi_j, \phi_i).$$

The following lemma is elementary.

**Lemma 5** For $v_h \in V_h$ with $v_h = \Sigma_{i=1}^s v_i^h \phi_i$, define $v^h = (v_1^h, v_2^h, \ldots, v_s^h)^T$ to be the corresponding vector in $\mathbb{R}^s$. Then

$$(\|v_h\|_2)^2 = (v^h)^T A_h^{-1} B_h A_h v^h.$$

By using the above lemma, it is elementary to verify the following two families of triangulations are regular: (a) one dimensional partitions; (b) triangulations generated by connecting the diagonal vertex in same direction of the squares in the finite difference mesh.

Now let us analyze the error after the correction step denoted by $e_2$. Then we have

$$e_2 = u_h - \bar{u}_h^2 = u_h - (\bar{u}_h^1 + e_2h) = e_h - e_2h.$$

The main result of this note is the following.
Theorem 6 Assume the sequence of triangulations \( \{ \mathcal{T}_k \} \) are regular. Let \( e_k \) and \( e_{2h} \) be the solutions of (6) and (7) respectively. Then there exist a constant \( C \) only depending on \( \beta, \gamma \) and regularity of \( \Omega \) such that

\[
\| e_h - e_{2h} \|_1 \leq C(\| f \|_0 + |\Psi| + |\tilde{u}_h|_2^2) h. \tag{12}
\]

Proof. Since the triangulations are regular, there exists a \( \tilde{u}^1 \in H^2(\Omega) \) such that

\[
\pi_h \tilde{u}^1 = \tilde{u}_h^1 \quad \text{and} \quad |\tilde{u}^1|_2 \leq \beta |\tilde{u}_h^1|_2.
\]

Set \( \tilde{K} = \{ \phi \in V : \phi \geq \Psi - \tilde{u}^1 \} \). Define \( e \in V \) such that

\[
\tilde{J}(e) = \inf_{\phi \in \tilde{K}} \tilde{J}(\phi).
\]

Then (6) and (7) are the discretizations of the above problem in \( h \)-mesh and \( 2h \)-mesh respectively. By applying Theorem 4, we have

\[
\| e_h - e \|_1 \leq C(\| r_h \|_0 + |\Psi - \tilde{u}^1|_2) h,
\]

\[
\| e_{2h} - e \|_1 \leq 2C(\| r_h \|_0 + |\Psi - \tilde{u}^1|_2) h.
\]

Hence

\[
\| e_h - e_{2h} \|_1 \leq 3C(\| r_h \|_0 + |\Psi - \tilde{u}^1|_2) h.
\]

By noticing that

\[
\| r_h \|_0 = \| f - A_h \tilde{u}_h^1 \|_0 \\
\leq \| f \|_0 + \| A_h \tilde{u}_h^1 \|_0 \\
= \| f \|_0 + \| \tilde{u}_h^1 \|_2^p
\]

and that

\[
|\Psi - \tilde{u}^1|_2 \leq |\Psi|_2 + |\tilde{u}^1|_2 \\
\leq |\Psi|_2 + \beta |\tilde{u}_h^1|_2,
\]

we have

\[
\| e_h - e_{2h} \|_1 \leq C(\| f \|_0 + |\Psi|_2 + |\tilde{u}_h^1|_2^p) h. \qed
\]

Theorem 6 suggests us that as long as \( |\tilde{u}_h^1|_2^p \) is \( O(1) \), then after the correction step the error achieves the discretization error. But \( |\tilde{u}_h^1|_2^p \) is computable and controllable.
References


