AMS 310.03 Midterm Examination  Fall, 2004

Name_________________________  ID_________________________  Signature_________________________

**Instructions.** Dear students: This is a close book and close notes exam. Anyone who cheats on the exam shall receive a score of 0 and a course grade of $F$. Please show complete procedures for full credit. You have the entire lecture to finish the exam. Please turn in this cover page with your solutions and staple them together. Please show your photo ID when submitting your exam. Good luck!

1. When the American League and the National League baseball champions are evenly matched, the probabilities that a World Series will end in 4, 5, 6, or 7 games are, respectively, $1/8$, $1/4$, $5/16$, and $5/16$. What is the expected length of a World Series when the two teams are evenly matched?
   Answer: HW 2 (#3.84).

2. A quality-control engineer inspects a random sample of 2 batteries from each lot of 10 car batteries ready to be shipped. If such a lot contains 3 batteries with defects, what is the probability that at least one bad battery will be selected?
   Answer:
   $$1 - \frac{3 \choose 0}{{10 \choose 2}} = 1 - \frac{1 * 7 * 6 / 2}{10 * 9 / 2} = \frac{8}{15} \approx 0.53$$

3. There are two identical cards that differ only in color. One card is colored blue on one side and red on the other. The other card is colored red on both sides. One card is chosen at random and flipped on the table. If the upper side of this card is red, what is the probability that its other side is also red?
   Answer: (Use Bayes’ Theorem) Let $RR$ denote the event that the chosen card is red on both sides and $RB$ denote the event that the chosen card is red on one side and blue on the other. Let $R$ be the event that the upper side of the chosen card is red, the desired probability is obtained by
   $$P(RR | R) = \frac{P(RR \cap R)}{P(R)} = \frac{P(R | RR)P(RR)}{P(R | RR)P(RR) + P(R | RB)P(RB)}$$
   $$= \frac{(1)(0.5)}{(1)(0.5) + (0.5)(0.5)} = \frac{2}{3} \approx 0.67$$

4. A judge is 60% sure that a suspect has committed a crime. During the course of the trial a witness convinces the judge that the criminal is left-handed. If 25% of the population is left-handed and the suspect is also left-handed, with this new information,
how certain should the judge be of the guilt of the suspect? 
Answer: (Use Bayes’ Theorem) Let $G$ and $\overline{C}$ denote, respectively, the events that suspect is or is not guilty. Let $L$ be the event that the suspect is left-handed, the desired probability is obtained by

$$P(G | L) = \frac{P(G \cap L)}{P(L)} = \frac{P(L \mid G) P(G)}{P(L \mid G) P(G) + P(L \mid \overline{G}) P(\overline{G})} = \frac{(1)(0.6)}{(1)(0.6) + (0.25)(1 - 0.6)} = 0.6 \approx 0.77$$

5. Suppose that an aircraft engine will fail in flight with probability $(1 - p)$ independent of the plane’s other engines. Also suppose that a plane can complete the journey successfully if at least half of its engines do not fail.

(a) Is it true that a 4-engine plane is always preferable to a 2-engine plane? Explain.

Answer: For a 2-engine plane, let $X$ be the number of engines that will NOT fail during the flight, then $X \sim Binomial (n = 2, p)$. Therefore

$$P(\text{a 2-engine plane will complete the journey successfully})$$

$$= P(X \geq 1) = 1 - P(X = 0)$$

$$= 1 - \binom{2}{0} p^0 (1-p)^{2-0} = 1 - (1-p)^2$$

Similarly, for a 4-engine plane, let $X$ be the number of engines that will NOT fail during the flight, then $X \sim Binomial (n = 4, p)$. Therefore

$$P(\text{a 4-engine plane will complete the journey successfully})$$

$$= P(X \geq 2) = 1 - P(X = 0) - P(X = 1)$$

$$= 1 - \binom{4}{0} p^0 (1-p)^{4-0} - \binom{4}{1} p^1 (1-p)^{4-1}$$

$$= 1 - (1-p)^4 - 4p(1-p)^3$$

For a 4-engine plane to be preferable to a 2-engine plane, we must have

$$1 - (1-p)^4 - 4p(1-p)^3 > 1 - (1-p)^2$$

A little simplification leads to

$$p > \frac{2}{3}$$

Therefore, we would prefer a 4-engine plane over a 2-engine plane if $p > \frac{2}{3}$ (or equivalently, $(1-p) < \frac{1}{3}$).
(b) For a successful journey, would you prefer a 4-engine plane or a 2-engine plane when \( p = 0.9 \)?
Answer: Since \( 0.9 > \frac{2}{3} \), we prefer a 4-engine plane over a 2-engine plane in this case. In fact, when \( p = 0.9 \), we have

\[
P(a \text{ 2-engine plane will complete the journey successfully}) = 1 - (1 - 0.9)^2 = 0.99,
\]

and

\[
P(a \text{ 4-engine plane will complete the journey successfully}) = 1 - (1 - 0.9)^4 - 4(0.9)(1 - 0.9)^3
= 1 - 0.0001 - 0.0036 = 0.9963
\]

Since 0.9963 > 0.99, we prefer a 4-engine plane over a 2-engine plane in this situation.

6. In a certain city during the month of May, the probability that a rainy day will be followed by another rainy day is 0.80 and the probability that a sunny day will be followed by a rainy day is 0.60. Assuming that each day is classified as being either rainy or sunny and that the weather on any given day depends only on the weather the day before, find the probability that in the given city a rainy day is followed by two more rainy days, then a sunny day, and finally another rainy day.
Answer: HW 2 (#3.70(b)).