Q1: $Z \sim N(0,1)$, and the value of random variable $W$ depends on a coin (fair coin) flip: $W = \begin{cases} Z, & \text{if head} \\ -Z, & \text{if tail} \end{cases}$ find the distribution of $W$. Is the joint distribution of $Z$ and $W$ a bivariate normal?

A1:

$$F_W(w) = P(W \leq w)$$
$$= P(W \leq w | H)P(H) + P(W \leq w | T)P(T)$$
$$= P(Z \leq w)P(H) + P(-Z \leq w)P(T)$$
$$= F_Z(w) \times \frac{1}{2} + P(Z \geq -w) \times \frac{1}{2}$$

$$= F_Z(w) \times \frac{1}{2} + F_Z(-w) \times \frac{1}{2}$$

So $W \sim N(0,1)$.

The joint distribution of $Z$ and $W$ is not normal. This can be shown by deriving the joint mgf of $Z$ and $W$ and compare it with the joint mgf of bivariate normal (which we already knew, from quiz 2).

$$M(t_1, t_2) = E(e^{t_1Z + t_2W})$$
$$= E(E(e^{t_1Z + t_2W} | \text{coin})))$$
$$= E(e^{t_1Z + t_2W} | H)P(H) + E(e^{t_1Z + t_2W} | T)P(T)$$
$$= E(e^{(t_1 + t_2)Z}) \times \frac{1}{2} + E(e^{(t_1 - t_2)Z}) \times \frac{1}{2}$$

$$= \frac{1}{2} (e^{\frac{(t_1 + t_2)^2}{2}} + e^{\frac{(t_1 - t_2)^2}{2}})$$

This is not the joint mgf of bivariate normal.