1. **Review of Probability, the Monty Hall Problem**

(http://en.wikipedia.org/wiki/Monty_Hall_problem)

The Monty Hall problem is a probability puzzle loosely based on the American television game show *Let’s Make a Deal* and named after the show's original host, Monty Hall.

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1 [but the door is not opened], and the host, who knows what’s behind the doors, opens another door (*always a door you did not choose, and behind which there is no car), say No. 3, which has a goat. He then says to you, "Do you want to switch (*i.e. pick door No. 2), or to stay (*i.e., stay with door No. 1 you picked initially)?" Is it to your advantage to switch your choice?

The answer will be clear by computing and comparing the following two probabilities: (1) what is your winning chance if your strategy is to stay? (2) what is your winning chance if your strategy is to switch?

Solutions: (*many possible ways – but not all of them are correct even if your answers are the right numbers. Here we present one approach using conditional probability.)

\[
P(\text{Win \_ By \_ Stay})
= P(\text{WBST \mid First \_ Door \_ Chosen \_ Has \_ Prize}) \cdot P(\text{F.D.C \_ Has \_ Prize})
+ P(\text{WBST \mid First \_ Door \_ Chosen \_ Has \_ No \_ Prize}) \cdot P(\text{F.D.C \_ Has \_ No \_ Prize})
\]

\[
= 1 \cdot \frac{1}{3} + 0 \cdot \frac{2}{3} = \frac{1}{3}
\]

\[
P(\text{Win \_ By \_ Switch})
= P(\text{WBSW \mid First \_ Door \_ Chosen \_ Has \_ Prize}) \cdot P(\text{F.D.C \_ Has \_ Prize})
+ P(\text{WBSW \mid First \_ Door \_ Chosen \_ Has \_ No \_ Prize}) \cdot P(\text{F.D.C \_ Has \_ No \_ Prize})
\]

\[
= 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}
\]

2. **Review of Probability (continued)**
**Exercise:**

A write to B and does not receive an answer. Assuming that one letter in n is lost in the mail, find the chance that B received the letter. It is to be assumed that B would have answered the letter if she had received it.

**Answer:**

We set event A and event B to be:

- A: person A got no answer;
- B: person B received the letter.

In the following, we apply

1. definition of conditional probability,
2. Bayes’ rule, and
3. the law of total probability respectively to obtain the answer.

\[
P(B|A) = \frac{P(B \cap A)}{P(A)}
= \frac{P(A|B)P(B)}{P(A)}
= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^C)P(B^C)}
= \frac{\frac{1}{n} \times \left(1 - \frac{1}{n}\right)}{\frac{1}{n} \times \left(1 - \frac{1}{n}\right) + 1 \times \frac{1}{n}}
= \frac{n - 1}{2n - 1}
\]


1. **Binomial distribution**

**Eg. 1.** Suppose each child’s birth will result in either a boy or a girl with equal probability. For a randomly selected family with 2 children, what is the chance that the chosen family has 1) 2 boys? 2) 2 girls? 3) a boy and a girl?

Solution: 25%; 25%; 50%

- \(P(B \text{ and } B) = P(B \cap B) = P(B) \cdot P(B) = 0.5 \cdot 0.5 = 0.25\)
- \(P(G \text{ and } G) = P(G \cap G) = P(G) \cdot P(G) = 0.5 \cdot 0.5 = 0.25\)
- \(P(B \text{ and } G) = P(B_1 \cap G_2 \text{ or } B_2 \cap G_1) = P((B_1 \cap G_2) \cup (B_2 \cap G_1))\)

**Binomial Experiment:**

1) It consists of n trials

2) Each trial results in 1 of 2 possible outcomes, “S” or “F”
3) The probability of getting a certain outcome, say "S", remains the same, from trial to trial, say \( P(\text{"S")}) = p \)

4) These trials are independent, that is the outcomes from the previous trials will not affect the outcomes of the upcoming trials.

**Eg. 1** (continued) \( n=2, \) let "S"=B, \( P(B)=0.5 \)

Let \( X \) denotes the total # of "S" among the \( n \) trials in a binomial experiment, then \( X \sim B(n, p) \), that is, Binomial Distribution with parameters \( n \) and \( p \). Its probability density function (pdf) is

\[
f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x=0,1,...,n
\]

**Here** \( \binom{n}{x} = \frac{n(n-1)...(n-x+1)}{x(x-1)...3\cdot 2\cdot 1} \); **(please note there are exactly \( x \) terms in the numerator, and in the denominator); for example, \( \binom{5}{2} = \frac{5 \cdot 4}{2 \cdot 1} = 10 \); also note: \( \binom{n}{x} = \binom{n}{n-x} \)

*** For a discrete random variable, its pdf is also called its probability mass function (pmf). For the above binomial pdf, we have:

\[
\sum_{i=0}^{n} P(X = x) = 1
\]

**Eg. 1** (continued) \( n=2, p=0.5, S=B, \) birth=trial
Answer: Let \( X \) denotes the total of boys from the 2 births. Then \( X \sim B(n=2, p=0.5) \)

1) \( P(2 \text{ boys})=P(X=2)=(\binom{2}{2})0.5^2(1-0.5)^{2-2}=.25 \)

2) \( P(2 \text{ girls})=P(X=0)=(\binom{2}{0})0.5^0(1-0.5)^{2-0}=.25 \)

3) \( P(1 \text{ boys and a girl})=P(X=1)=(\binom{2}{1})0.5^1(1-0.5)^{2-1}=.5 \)

4) What is the probability of having at least 1 boy?

\[ P(\text{at least 1 boy})=P(X \geq 1)=P(X=1)+P(X=2) = .5+.25 = .75 \]

**Eg. 2.** An exam consists of 10 multiple choice questions. Each question has 4 possible choices. Only 1 is correct. Jeff did not study for the exam. So he just guesses at the right answer for each question (pure guess, not an educated guess). What is his chance of passing the exam? That is, to make at least 6 correct answers.

**Answer:** Yes, this is a binomial experiment with \( n=10, p=0.25 \), "S"=choose the right answer for each question.

Let \( X \) be the total # of "S"

\[
P(\text{pass})=P(X \geq 6)=P(X=6 \text{ or } X=7 \text{ or } X=8 \text{ or } X=9 \text{ or } X=10)
= P(X=6)+ P(X=7)+ P(X=8)+ P(X=9)+ P(X=10)
=\binom{10}{6}0.25^6(1-0.5)^4 + ...
\]

**(2) Bernoulli Distribution**
• X~Bernoulli(p). It can take on two possible values, say success (S) or failure (F) with probability p and (1-p) respectively.
• That is:
  \[ P(X = 'S') = p; \ P(X = 'F') = 1 - p \]
• Let X = number of "S", then \( X = \begin{cases} 0, & 1 - p \\ 1, & p \end{cases} \)

The pdf of X can be written as:
\[ f(x) = P(X = x) = p^x(1 - p)^{1-x} ; x = 0, 1 \]

**Relation between Bernoulli RV and Binomial RV.**

(1) X~Bernoulli(p) => it is indeed a special case of Binomial random variable when \( n = 1 \) (*only one trial), that is:
\[ B(n = 1, p) \]

(2) Let \( X_i \sim \text{Bernoulli}(p), i = 1, \ldots, n \). Furthermore, \( X_i \)'s are all independent. Let \( X = \sum_{i=1}^{n} X_i \). Then, \( X \sim B(n, p) \) (**Exercise, prove this!). **Note:** *** This links directly to the Binomial Experiment with \( X_i \) denotes the number of ‘S’ for the \( i^{th} \) trial.

Homework 1 (Due Tuesday, February 3, before class):

1. Jake has been caught stealing cattle, and is brought into town for justice. The judge is his ex-wife Gretchen, who wants to show him some sympathy, but the law clearly calls for two shots to be taken at Jake from close range. To make things a little better for Jake, Gretchen tells him she will place two bullets into a six-chambered revolver in successive order. She will spin the chamber, close it, and take one shot. If Jake is still alive, she will then either take another shot, or spin the chamber again before shooting.

   Jake is a bit incredulous that his own ex-wife would carry out the punishment, and a bit sad that she was always such a rule follower. He steels himself as Gretchen loads the chambers, spins the revolver, and pulls the trigger. Whew! It was blank. Then Gretchen asks, "Do you want me to pull the trigger again, or should I spin the chamber a second time before pulling the trigger?"

   What should Jake choose?
   (http://www.braingle.com/Probability.html)

2. Review Chapters 1, 2, 3

   In particular, please
   (1) review the normal distribution and the binomial distribution,
   (2) review definitions of the probability density function, cumulative distribution function, and mathematical expectations.