Lecture 10.

Today: Cramer-Rao lower bound

Unbiased Estimator of $\theta$, say $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, ...$

It could be really difficult for us to compare $\text{Var}(\hat{\theta}_i)$ when there are many of them.

**Theorem. Cramer-Rao Lower Bound**

Let $Y_1, Y_2, ..., Y_n$ be a random sample from a population with p.d.f. $f(y; \theta)$. Let $\hat{\theta} = h(y_1, y_2, ..., y_n)$ be an unbiased estimator of $\theta$.

Given some regularity conditions (continuous differentiable etc.) and the domain of $f(y_i; \theta)$, does not depend on $\theta$. Then, we have

$$\text{Var}(\hat{\theta}) \geq \frac{1}{n \text{E}[ (\frac{\partial \ln f(\theta)}{\partial \theta})^2 ]} \geq \frac{1}{n \text{E}[ (\frac{\partial^2 \ln f(\theta)}{\partial \theta^2 })]$$

**Theorem. Properties of the MLE**

Let $Y_i \overset{i.i.d.}{\sim} f(y; \theta), i = 1, 2, ..., n$

Let $\hat{\theta}$ be the MLE of $\theta$, then

$$\hat{\theta} \overset{n \to \infty}{\rightarrow} N \left( \theta, \frac{1}{n \text{E}[ (\frac{\partial \ln f(\theta)}{\partial \theta})^2 ]} \right)$$

The MLE is asymptotically unbiased and its asymptotic variance: C-R lower bound

**Definition. Fisher Information.**

$$I(\theta) = \text{E} \left[ \left( \frac{\partial \ln f(\theta)}{\partial \theta} \right)^2 \right] = -\text{E} \left[ \frac{\partial^2 \ln [f(x; \theta)]}{\partial \theta^2} \right]$$

**Definition. The Efficiency – of an unbiased estimator $Y$ is:**

$$e(Y) = \frac{CRLB}{\text{Var}(Y)}$$
Harald Cramér (above) was born in Stockholm, Sweden on September 25, 1893, and died there on October 25, 1985. (wiki). Calyampudi Radhakrishna Rao (below), FRS known as C R Rao (born 10 September 1920) is an Indian statistician. He is professor emeritus at Penn State University and Research Professor at the University at Buffalo. Rao was awarded the US National Medal of Science in 2002. (wiki)
Example 1. Let $Y_1, Y_2, \ldots, Y_n \sim \text{i.i.d.}\ Bernoulli(p)$

1. MLE of $p$?
2. What are the mean and variance of the MLE of $p$?
3. What is the Cramer-Rao lower bound for an unbiased estimator of $p$?

Solution. $P(Y = y) = f(y; p) = p^y (1 - p)^{1-y}, y = 0, 1$

1. 
   
   $L = \prod_{i=1}^{n} f(y_i; p) = \prod_{i=1}^{n} [p^y_i (1 - p)^{1-y_i}] = p^{\sum y_i} (1 - p)^{n - \sum y_i}$

   $l = \ln L = (\sum y_i) \ln p + (n - \sum y_i) \ln (1 - p)$

   Solving:

   $\frac{dl}{dp} = \frac{\sum y_i}{p} - \frac{n - \sum y_i}{1 - p} = 0,$

   we have the MLE:

   $\hat{p} = \frac{\sum_{i=1}^{n} y_i}{n}$

2. 

   $E(\hat{p}) = p, Var(\hat{p}) = \frac{p(1 - p)}{n}$

3. 

   $\ln f(y, p) = y \ln p + (1 - y) \ln (1 - p)$

   $\frac{\partial \ln f(y, p)}{\partial p} = \frac{y}{p} - \frac{1 - y}{1 - p}$

   $\frac{\partial^2 \ln f(y, p)}{\partial p^2} = -\frac{y}{p^2} - \frac{1 - y}{(1 - p)^2}$

   $E \left[ -\frac{Y}{p^2} - \frac{1 - Y}{(1 - p)^2} \right] = -\frac{1}{p(1 - p)}$

   C-R lower bound

   $\text{Var}(\hat{p}) \geq \left\{ -nE[\frac{\partial^2 \ln f(y, p)}{\partial p^2}] \right\}^{-1} = \frac{p(1 - p)}{n}$
Thus, the MLE of $p$ is unbiased and its variance = C-R lower bound.

**Definition. Efficient Estimator**

If $\hat{\theta}$ is an unbiased estimator of $\theta$ and its variance = C-R lower bound, then $\hat{\theta}$ is an efficient estimator of $\theta$.

**Definition. Best Estimator (UMVUE, or, MVUE)**

If $\hat{\theta}$ is an unbiased estimator of $\theta$ and $\text{var}(\hat{\theta}) \leq \text{var}(\tilde{\theta})$ for all unbiased estimator $\tilde{\theta}$, then $\hat{\theta}$ is a best estimator for $\theta$.

The best estimator is also called the uniformly minimum variance unbiased estimator (UMVUE), or simply, the minimum variance unbiased estimator (MVUE).

\[
\begin{align*}
\text{Always true} & \quad \Rightarrow \quad \text{Efficient Estimator} \\
\text{May not be true} & \quad \Rightarrow \quad \text{Best Estimator}
\end{align*}
\]

**Example 2.** If $Y_1, Y_2, \ldots, Y_n$ is a random sample from $f(y; \theta) = \frac{2y}{\theta^2}, 0 < y < \theta$, then $\tilde{\theta} = \frac{3}{2} \bar{Y}$ is an unbiased estimator for $\theta$.

Compute 1. $\text{Var}(\hat{\theta})$ and 2. C-R lower bound for $f_Y(y; \theta)$

**Solution.**

1. \[
\text{Var}(\tilde{\theta}) = \text{Var}\left(\frac{3}{2} \bar{Y}\right) = \frac{9}{4} \text{Var}\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{9}{4n^2} \sum_{i=1}^{n} \text{Var}(Y_i)
\]

\[
\text{Var}(Y_i) = E(Y_i^2) - [E(Y_i)]^2 = \int_{0}^{\theta} y^2 \frac{2y}{\theta^2} dy - \left[\int_{0}^{\theta} y^2 \frac{2y}{\theta^2} dy\right]^2 = \frac{\theta^2}{18}
\]

Therefore, $\text{Var}(\tilde{\theta}) = \frac{9}{4n^2} \frac{n\theta^2}{18} = \frac{\theta^2}{8n}$

2. C-R lower bound

\[
\ln f_Y(y; \theta) = \ln\left(\frac{2y}{\theta^2}\right) = \ln 2y - 2\ln \theta
\]
\[ \frac{\partial \ln f_y(y; \theta)}{\partial \theta} = -\frac{2}{\theta} \]
\[ E[(\frac{\partial \ln f_y(y; \theta)^2}{\partial \theta})] = E(\frac{4}{\theta^2}) = \int_0^\theta \frac{4 \cdot 2y}{\theta^2} \, dy = \frac{4}{\theta^2} \]
\[ \frac{1}{nE[(\frac{\partial \ln f_y(y; \theta)^2}{\partial \theta})]} = \frac{\theta^2}{4n} \]

The value of 1 is less than the value of 2. But, it is NOT a contradiction to the theorem. Because the domain, \(0 < y < \theta\), depends on \(\theta\). Thus the C-R Theorem doesn’t hold for this problem.