What is a matrix?

A rectangular array of numbers (we will concentrate on real numbers). A nxm matrix has ‘n’ rows and ‘m’ columns.

\[ M_{3 \times 4} = \begin{bmatrix}
M_{11} & M_{12} & M_{13} & M_{14} \\
M_{21} & M_{22} & M_{23} & M_{24} \\
M_{31} & M_{32} & M_{33} & M_{34}
\end{bmatrix} \]

- First row
- Second row
- Third row
- Fourth column
- First column
- Second column
- Third column
- Fourth column

Row number

Column number
A vector is an array of ‘n’ numbers

A row vector of length ‘n’ is a 1xn matrix

\[
\begin{bmatrix}
  a_1 & a_2 & a_3 & a_4 \\
\end{bmatrix}
\]

A column vector of length ‘m’ is a mx1 matrix

\[
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
\end{bmatrix}
\]
Special matrices

Zero matrix: A matrix all of whose entries are zero

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Identity matrix: A square matrix which has ‘1’ s on the diagonal and zeros everywhere else.

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Matrix operations

Equality of matrices

If $A$ and $B$ are two matrices of the same size, then they are “equal” if each and every entry of one matrix equals the corresponding entry of the other.

\[
A = \begin{bmatrix}
1 & 2 & 4 \\
-3 & 0 & 7 \\
9 & 1 & 5
\end{bmatrix}, \quad B = \begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix}
\]

$a = 1, \quad b = 2, \quad c = 4,$

$A = B \iff d = -3, \quad e = 0, \quad f = 7,$

$g = 9, \quad h = 1, \quad i = 5.$
If $\mathbf{A}$ and $\mathbf{B}$ are two matrices of the same size, then the sum of the matrices is a matrix $\mathbf{C}=\mathbf{A}+\mathbf{B}$ whose entries are the sums of the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$.

\[
\mathbf{A} = \begin{bmatrix}
1 & 2 & 4 \\
-3 & 0 & 7 \\
9 & 1 & 5 
\end{bmatrix}, \quad
\mathbf{B} = \begin{bmatrix}
-1 & 3 & 10 \\
-3 & 1 & 0 \\
1 & 0 & 6 
\end{bmatrix}
\]

\[
\mathbf{C} = \mathbf{A} + \mathbf{B} = \begin{bmatrix}
0 & 5 & 14 \\
-6 & 1 & 7 \\
10 & 1 & 11 
\end{bmatrix}
\]
Properties of matrix addition:
1. Matrix addition is **commutative** (order of addition does not matter)
   \[ A + B = B + A \]
2. Matrix addition is **associative**
   \[ A + (B + C) = (A + B) + C \]
3. Addition of the zero matrix
   \[ A + 0 = 0 + A = A \]
If $\mathbf{A}$ is a matrix and $c$ is a scalar, then the product $c\mathbf{A}$ is a matrix whose entries are obtained by multiplying each of the entries of $\mathbf{A}$ by $c$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & 7 \\ 9 & 1 & 5 \end{bmatrix} \quad c = 3$$

$$c\mathbf{A} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & 21 \\ 27 & 3 & 15 \end{bmatrix}$$
Matrix operations

Multiplication by a scalar

If \( A \) is a matrix and \( c = -1 \) is a scalar, then the product \((-1)A = -A\) is a matrix whose entries are obtained by multiplying each of the entries of \( A \) by \(-1\)

\[
A = \begin{bmatrix}
1 & 2 & 4 \\
-3 & 0 & 7 \\
9 & 1 & 5
\end{bmatrix}
\]

\[
c = -1
\]

\[
cA = -A = \begin{bmatrix}
-1 & -2 & -4 \\
3 & 0 & -7 \\
-9 & -1 & -5
\end{bmatrix}
\]
If \( \mathbf{A} \) and \( \mathbf{B} \) are two square matrices of the same size, then \( \mathbf{A-B} \) is defined as the sum \( \mathbf{A} + (-1)\mathbf{B} \).

\[
\mathbf{A} = \begin{bmatrix}
1 & 2 & 4 \\
-3 & 0 & 7 \\
9 & 1 & 5 \\
\end{bmatrix}
\quad
\mathbf{B} = \begin{bmatrix}
-1 & 3 & 10 \\
-3 & 1 & 0 \\
1 & 0 & 6 \\
\end{bmatrix}
\]

\[
\mathbf{C} = \mathbf{A} - \mathbf{B} = \begin{bmatrix}
2 & -1 & -6 \\
0 & -1 & 7 \\
8 & 1 & -1 \\
\end{bmatrix}
\]

Note that \( \mathbf{A} - \mathbf{A} = \mathbf{0} \) and \( \mathbf{0} - \mathbf{A} = -\mathbf{A} \).
If $A$ is a $mxn$ matrix, then the transpose of $A$ is the $nxm$ matrix whose first column is the first row of $A$, whose second column is the second column of $A$ and so on.

\[
A = \begin{bmatrix}
1 & 2 & 4 \\
-3 & 0 & 7 \\
9 & 1 & 5
\end{bmatrix} \quad \iff \quad A^T = \begin{bmatrix}
1 & -3 & 9 \\
2 & 0 & 1 \\
4 & 7 & 5
\end{bmatrix}
\]
If $A$ is a square matrix ($mxm$), it is called symmetric if

$$A = A^T$$
If \( \mathbf{a} \) and \( \mathbf{b} \) are two vectors of the same size

\[
\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
\]

The scalar (dot) product of \( \mathbf{a} \) and \( \mathbf{b} \) is a scalar obtained by adding the products of corresponding entries of the two vectors

\[
\mathbf{a}^\mathsf{T}\mathbf{b} = (a_1 b_1 + a_2 b_2 + a_3 b_3)
\]
For a product to be defined, the number of columns of $\mathbf{A}$ must be equal to the number of rows of $\mathbf{B}$.
If \( \mathbf{A} \) is a \( m \times r \) matrix and \( \mathbf{B} \) is a \( r \times n \) matrix, then the product \( \mathbf{C} = \mathbf{A} \mathbf{B} \) is a \( m \times n \) matrix whose entries are obtained as follows. The entry corresponding to row ‘\( i \)’ and column ‘\( j \)’ of \( \mathbf{C} \) is the dot product of the vectors formed by the row ‘\( i \)’ of \( \mathbf{A} \) and column ‘\( j \)’ of \( \mathbf{B} \)

\[
\mathbf{A}_{3x3} = \begin{bmatrix}
1 & 2 & 4 \\
-3 & 0 & 7 \\
9 & 1 & 5
\end{bmatrix}
\quad \mathbf{B}_{3x2} = \begin{bmatrix}
-1 & 3 \\
-3 & 1 \\
1 & 0
\end{bmatrix}
\]

\[
\mathbf{C}_{3x2} = \mathbf{A} \mathbf{B} = \begin{bmatrix}
-3 & 5 \\
10 & -9 \\
-7 & 28
\end{bmatrix}
\]

Notice \( \begin{bmatrix} 1 & -1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix}^T \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = -3 \)
Properties of matrix multiplication:

1. Matrix multiplication is **noncommutative** (order of addition **does** matter)

\[ \text{AB} \neq \text{BA in general} \]

- It may be that the product \( \text{AB} \) exists but \( \text{BA} \) does not (e.g. in the previous example \( \text{C}=\text{AB} \) is a 3x2 matrix, but \( \text{BA} \) does not exist)

- Even if the product exists, the products \( \text{AB} \) and \( \text{BA} \) are not generally the same
2. Matrix multiplication is **associative**

\[ A(BC) = (AB)C \]

3. **Distributive law**

\[ A(B + C) = AB + AC \]

\[ (B + C)A = BA + CA \]

4. **Multiplication by identity matrix**

\[ AI = A; IA = A \]

5. **Multiplication by zero matrix** \[ A0 = 0; 0A = 0 \]

6. \[ (AB)^T = B^T A^T \]
1. If \( \mathbf{A} \), \( \mathbf{B} \) and \( \mathbf{C} \) are square matrices of the same size, and \( \mathbf{A} \neq 0 \) then \( \mathbf{AB} = \mathbf{AC} \) does not necessarily mean that \( \mathbf{B} = \mathbf{C} \).

2. \( \mathbf{AB} = 0 \) does not necessarily imply that either \( \mathbf{A} \) or \( \mathbf{B} \) is zero.
If \( \mathbf{A} \) is any square matrix and \( \mathbf{B} \) is another square matrix satisfying the conditions

\[
\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A} = \mathbf{I}
\]

Then

(a) The matrix \( \mathbf{A} \) is called invertible, and
(b) the matrix \( \mathbf{B} \) is the inverse of \( \mathbf{A} \) and is denoted as \( \mathbf{A}^{-1} \).

The inverse of a matrix is unique.
The inverse of a matrix is **unique**
Assume that \( B \) and \( C \) both are inverses of \( A \)

\[
AB = BA = I \\
AC = CA = I \\
(BA)C = IC = C \\
B(AC) = BI = B
\]

\( \therefore B = C \)

Hence a matrix **cannot** have two or more inverses.
Inverse of a matrix

Property 1: If $A$ is any invertible square matrix the inverse of its inverse is the matrix $A$ itself

$$
\left( A^{-1} \right)^{-1} = A
$$

Property 2: If $A$ is any invertible square matrix and $k$ is any scalar then

$$
\left( kA \right)^{-1} = \frac{1}{k} A^{-1}
$$
Inverse of a matrix

Property 3: If $A$ and $B$ are invertible square matrices then

$$(AB)(AB)^{-1} = I$$

Premultiplying both sides by $A^{-1}$

$$A^{-1}(AB)(AB)^{-1} = A^{-1}$$

$$(A^{-1}A)B(AB)^{-1} = A^{-1}$$

$$B(AB)^{-1} = A^{-1}$$

Premultiplying both sides by $B^{-1}$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Properties
What is a determinant?

The **determinant** of a **square matrix** is a number obtained in a specific manner from the matrix.

For a 1x1 matrix:

\[
A = \begin{bmatrix} a_{11} \end{bmatrix}; \quad \text{det}(A) = a_{11}
\]

For a 2x2 matrix:

\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; \quad \text{det}(A) = a_{11}a_{22} - a_{12}a_{21}
\]

Product along **red arrow** minus product along **blue arrow**
Consider the matrix \( A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} \)

Notice (1) A matrix is an array of numbers
(2) A matrix is enclosed by square brackets

\[
\text{det}(A) = \begin{vmatrix} 1 & 3 \\ 5 & 7 \end{vmatrix} = 1 \times 7 - 3 \times 5 = -8
\]

Notice (1) The determinant of a matrix is a number
(2) The symbol for the determinant of a matrix is a pair of parallel lines
Duplicate column method for 3x3 matrix

For **ONLY** a 3x3 matrix write down the first two columns after the third column

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

Sum of products along **red arrow** minus sum of products along **blue arrow**

\[
\text{det}(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\
- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}
\]

This technique works only for 3x3 matrices
Example 2

\[ A = \begin{bmatrix} 2 & 4 & -3 \\ 1 & 0 & 4 \\ 2 & -1 & 2 \end{bmatrix} \]

Sum of red terms = 0 + 32 + 3 = 35
Sum of blue terms = 0 − 8 + 8 = 0
Determinant of matrix \( A \)= \( \det(A) = 35 − 0 = 35 \)
Finding determinant using inspection

Special case. If two rows or two columns are proportional (i.e. multiples of each other), then the determinant of the matrix is zero

\[
\begin{vmatrix}
2 & 7 & 8 \\
3 & 2 & 4 \\
-2 & -7 & -8
\end{vmatrix} = 0
\]

because rows 1 and 3 are proportional to each other

If the determinant of a matrix is zero, it is called a singular matrix
Cofactor method

If $A$ is a square matrix

$$A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}$$

The **minor**, $M_{ij}$, of entry $a_{ij}$ is the determinant of the submatrix that remains after the $i^{th}$ row and $j^{th}$ column are deleted from $A$. The **cofactor** of entry $a_{ij}$ is $C_{ij}=(-1)^{(i+j)} M_{ij}$

$$M_{12} = \begin{vmatrix}
  a_{21} & a_{23} \\
  a_{31} & a_{33}
\end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$$

$$C_{12} = -M_{12} = -\begin{vmatrix}
  a_{21} & a_{23} \\
  a_{31} & a_{33}
\end{vmatrix}$$
What is a cofactor?

Sign of cofactor

\[
\begin{array}{ccc}
+ & - & + \\
- & + & - \\
+ & - & + \\
\end{array}
\]

Find the minor and cofactor of \(a_{33}\)

\[
A = \begin{bmatrix}
2 & 4 & -3 \\
1 & 0 & 4 \\
2 & -1 & 2 \\
\end{bmatrix}
\]

Minor

\[
M_{33} = \begin{vmatrix}
2 & 4 \\
1 & 0 \\
\end{vmatrix} = 2 \times 0 - 4 \times 1 = -4
\]

Cofactor

\[
C_{33} = (-1)^{3+3} M_{33} = M_{33} = -4
\]
The determinant of a \( n \times n \) matrix \( A \) can be computed by multiplying \textbf{ALL the entries in ANY row (or column)} by their \textit{cofactors} and \textbf{adding} the resulting products. That is, for each \( 1 \leq i \leq n \) and \( 1 \leq j \leq n \):

\[
\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}
\]

Cofactor expansion along the \( j^{th} \) column

Cofactor expansion along the \( i^{th} \) row

\[
\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}
\]
### Example 3: evaluate $\det(A)$ for:

$$A = \begin{bmatrix}
1 & 0 & 2 & -3 \\
3 & 4 & 0 & 1 \\
-1 & 5 & 2 & -2 \\
0 & 1 & 1 & 3
\end{bmatrix}$$

$$\det(A) = \begin{vmatrix}
4 & 0 & 1 \\
5 & 2 & -2 \\
1 & 1 & 3 \\
3 & 4 & 0
\end{vmatrix} - \begin{vmatrix}
\begin{array}{ccc}
3 & 0 & 1 \\
-1 & 2 & -2 \\
0 & 1 & 3 \\
0 & 1 & 3
\end{array}
\end{vmatrix} + \begin{vmatrix}
\begin{array}{ccc}
3 & 4 & 1 \\
-1 & 5 & -2 \\
0 & 1 & 3 \\
0 & 1 & 3
\end{array}
\end{vmatrix}$$

$$\det(A) = (1)(35) - 0 + (2)(62) - (-3)(13) = 198$$
Example 4: evaluate

\[
\begin{vmatrix}
1 & 5 & -3 \\
1 & 0 & 2 \\
3 & -1 & 2 \\
\end{vmatrix}
\]

By a cofactor along the third column

\[
\text{det}(A) = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}
\]

\[
\text{det}(A) = (-1)^4 \begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} + (-1)^5 \begin{vmatrix} 1 & 5 \\ 3 & -1 \end{vmatrix} + (-1)^6 \begin{vmatrix} 1 & 5 \\ 1 & 0 \end{vmatrix}
\]

\[
= \text{det}(A) = -3(-1) + 2(-1)^5(-1-15) + 2(0-5) = 25
\]
Quadratic form

The scalar

\[ U = d^T k d \]

\( d \) = vector

\( k \) = square matrix

Is known as a **quadratic form**

If \( U > 0 \): Matrix \( k \) is known as **positive definite**
If \( U \geq 0 \): Matrix \( k \) is known as **positive semidefinite**
Let \[ \underline{d} = \begin{cases} d_1 \\ d_2 \end{cases} \quad \underline{k} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \]

Then

\[ U = \underline{d}^T \underline{k} \underline{d} = \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \]

\[ = \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} k_{11}d_1 + k_{12}d_2 \\ k_{12}d_1 + k_{22}d_2 \end{bmatrix} \]

\[ = d_1(k_{11}d_1 + k_{12}d_2) + d_2(k_{12}d_1 + k_{22}d_2) \]

\[ = k_{11}d_1^2 + 2k_{12}d_1d_2 + k_{22}d_2^2 \]
Differentiation of quadratic form

Differentiate $U$ wrt $d_1$

$$\frac{\partial U}{\partial d_1} = 2k_{11}d_1 + 2k_{12}d_2$$

Differentiate $U$ wrt $d_2$

$$\frac{\partial U}{\partial d_2} = 2k_{12}d_1 + 2k_{22}d_2$$
Hence

\[
\frac{\partial U}{\partial d} \equiv \begin{bmatrix}
\frac{\partial U}{\partial d_1} \\
\frac{\partial U}{\partial d_2}
\end{bmatrix} = 2 \begin{bmatrix}
k_{11} & k_{12} \\
k_{12} & k_{22}
\end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}
\]

\[
= 2 k d
\]
Review:

• The Derivatives of Vector Functions

• The Chain Rule for Vector Functions
1. The Derivatives of Vector Functions

Let \( \mathbf{x} \) and \( \mathbf{y} \) be vectors of orders \( n \) and \( m \) respectively:

\[
\mathbf{x} = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_m
\end{bmatrix},
\]

where each component \( y_i \) may be a function of all the \( x_j \), a fact represented by saying that \( \mathbf{y} \) is a function of \( \mathbf{x} \), or

\[
\mathbf{y} = \mathbf{y}(\mathbf{x}).
\]

If \( n = 1 \), \( \mathbf{x} \) reduces to a scalar, which we call \( x \). If \( m = 1 \), \( \mathbf{y} \) reduces to a scalar, which we call \( y \). Various applications are studied in the following subsections.
1.1 Derivative of Vector with Respect to Vector

The derivative of the vector $\mathbf{y}$ with respect to vector $\mathbf{x}$ is the $n \times m$ matrix

$$
\frac{\partial \mathbf{y}}{\partial \mathbf{x}} \overset{\text{def}}{=} \begin{bmatrix}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_1} \\
\frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \cdots & \frac{\partial y_m}{\partial x_n}
\end{bmatrix}
$$
1.2 Derivative of a Scalar with Respect to Vector

If \( y \) is a scalar

\[
\frac{\partial y}{\partial x} \overset{\text{def}}{=} \begin{bmatrix}
\frac{\partial y}{\partial x_1} \\
\frac{\partial y}{\partial x_2} \\
\vdots \\
\frac{\partial y}{\partial x_n}
\end{bmatrix}.
\]

It is also called the gradient of \( y \) with respect to a vector variable \( x \), denoted by \( \nabla y \).

1.3 Derivative of a Vector with Respect to Scalar

\[
\frac{\partial y}{\partial x} \overset{\text{def}}{=} \begin{bmatrix}
\frac{\partial y_1}{\partial x} \\
\frac{\partial y_2}{\partial x} \\
\vdots \\
\frac{\partial y_m}{\partial x}
\end{bmatrix}
\]
Example 5

Given

\[ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \]

and

\[ y_1 = x_1^2 - x_2 \]
\[ y_2 = x_3^2 + 3x_2 \]

the partial derivative matrix \( \frac{\partial y}{\partial x} \) is computed as follows:

\[
\frac{\partial y}{\partial x} = \begin{bmatrix}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} \\
\frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} \\
\frac{\partial y_1}{\partial x_3} & \frac{\partial y_2}{\partial x_3}
\end{bmatrix} = \begin{bmatrix}
2x_1 & 0 \\
-1 & 3 \\
0 & 2x_3
\end{bmatrix}
\]
In Matlab

```matlab
>> syms x1 x2 x3 real;
>> y1=x1^2-x2;
>> y2=x3^2+3*x2;
>> J = jacobian([y1;y2], [x1 x2 x3])
J =
[ 2*x1, -1, 0]
[ 0, 3, 2*x3]
Note: Matlab defines the derivatives as the transposes of those given in this lecture.
>> J'
ans =
[ 2*x1, 0]
[ -1, 3]
[ 0, 2*x3]
```
Some useful vector derivative formulas

\[ \frac{\partial}{\partial \mathbf{x}} \mathbf{C} \mathbf{x} = \mathbf{C}^T \]

\[ \frac{\partial}{\partial \mathbf{x}} \mathbf{x}^T \mathbf{C} \]

\[ \frac{\partial}{\partial \mathbf{x}} \mathbf{x}^T \mathbf{x} = 2\mathbf{x} \]

Exercise:
Important Property of Quadratic Form $\mathbf{x}^T \mathbf{C} \mathbf{x}$

\[
\frac{\partial}{\partial \mathbf{x}} (\mathbf{x}^T \mathbf{C} \mathbf{x}) = (\mathbf{C} + \mathbf{C}^T) \mathbf{x}
\]

**Proof:**

\[
\mathbf{x}^T \mathbf{C} \mathbf{x} = \sum_{i=1}^{n} \left[ x_i \sum_{j=1}^{n} (x_j C_{ij}) \right]
\]

\[
\Rightarrow \frac{\partial (\mathbf{x}^T \mathbf{C} \mathbf{x})}{\partial x_k} = \frac{\partial \left\{ \sum_{i=1}^{n} \left[ x_i \sum_{j=1}^{n} (x_j C_{ij}) \right] \right\}}{\partial x_k} = \frac{\partial \left\{ \sum_{j=1}^{n} (x_j C_{kj}) \right\}}{\partial x_k} + \frac{\partial \left\{ \sum_{i=1}^{n} [x_i x_k C_{ik}] \right\}}{\partial x_k}
\]

\[
= \sum_{j=1}^{n} x_j C_{kj} + \sum_{i=1}^{n} x_i C_{ik}
\]

\[
\Rightarrow \frac{\partial (\mathbf{x}^T \mathbf{C} \mathbf{x})}{\partial \mathbf{x}} = \mathbf{C} \mathbf{x} + \mathbf{C}^T \mathbf{x} = (\mathbf{C} + \mathbf{C}^T) \mathbf{x}
\]

**If C is symmetric:**

\[
\frac{\partial}{\partial \mathbf{x}} (\mathbf{x}^T \mathbf{C} \mathbf{x}) = 2 \mathbf{C} \mathbf{x}
\]
2. The Chain Rule for Vector Functions

Let

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix} \quad \text{and} \quad \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}
\]

where \( z \) is a function of \( y \), which is in turn a function of \( x \), we can write

\[
\left( \frac{\partial \mathbf{z}}{\partial \mathbf{x}} \right)^T = \begin{bmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \cdots & \frac{\partial z_1}{\partial x_n} \\ \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial x_2} & \cdots & \frac{\partial z_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial z_m}{\partial x_1} & \frac{\partial z_m}{\partial x_2} & \cdots & \frac{\partial z_m}{\partial x_n} \end{bmatrix}
\]

Each entry of this matrix may be expanded as

\[
\frac{\partial z_i}{\partial x_j} = \sum_{q=1}^r \frac{\partial z_i}{\partial y_q} \frac{\partial y_q}{\partial x_j} \quad \left\{ \begin{array}{l} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n. \end{array} \right.
\]
Then

\[
\left( \frac{\partial z}{\partial x} \right)^T = \begin{bmatrix}
\sum \frac{\partial z_1}{\partial y_q} \frac{\partial y_q}{\partial x_1} & \sum \frac{\partial z_1}{\partial y_q} \frac{\partial y_q}{\partial x_2} & \cdots & \sum \frac{\partial z_1}{\partial y_q} \frac{\partial y_q}{\partial x_n} \\
\sum \frac{\partial z_2}{\partial y_q} \frac{\partial y_q}{\partial x_1} & \sum \frac{\partial z_2}{\partial y_q} \frac{\partial y_q}{\partial x_2} & \cdots & \sum \frac{\partial z_2}{\partial y_q} \frac{\partial y_q}{\partial x_n} \\
\vdots & \vdots & & \vdots \\
\sum \frac{\partial z_m}{\partial y_q} \frac{\partial y_q}{\partial x_1} & \sum \frac{\partial z_m}{\partial y_q} \frac{\partial y_q}{\partial x_2} & \cdots & \sum \frac{\partial z_m}{\partial y_q} \frac{\partial y_q}{\partial x_n}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{\partial z_1}{\partial y_1} & \frac{\partial z_1}{\partial y_2} & \cdots & \frac{\partial z_1}{\partial y_r} \\
\frac{\partial z_2}{\partial y_1} & \frac{\partial z_2}{\partial y_2} & \cdots & \frac{\partial z_2}{\partial y_r} \\
\vdots & \vdots & & \vdots \\
\frac{\partial z_m}{\partial y_1} & \frac{\partial z_m}{\partial y_2} & \cdots & \frac{\partial z_m}{\partial y_r}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\
\frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\
\vdots & \vdots & & \vdots \\
\frac{\partial y_r}{\partial x_1} & \frac{\partial y_r}{\partial x_2} & \cdots & \frac{\partial y_r}{\partial x_n}
\end{bmatrix}
\]

\[
= \left( \frac{\partial z}{\partial y} \right)^T \left( \frac{\partial y}{\partial x} \right)^T = \left( \frac{\partial y}{\partial x} \frac{\partial z}{\partial y} \right)^T.
\]

On transposing both sides, we finally obtain

\[
\frac{\partial z}{\partial x} = \frac{\partial y}{\partial x} \frac{\partial z}{\partial y},
\]

This is the chain rule for vectors (different from the conventional chain rule of calculus, the chain of matrices builds toward the left).
x, y are as in Example 1 and z is a function of y defined as

$$\begin{align*}
z &= \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}, \quad \text{and} \quad \begin{cases}
z_1 &= y_1^2 - 2y_2 \\
z_2 &= y_2^2 - y_1 \\
z_3 &= y_1^2 + y_2^2 \\
z_4 &= 2y_1 + y_2
\end{cases}, \quad \text{we have}
\end{align*}$$

$$\frac{\partial z}{\partial y} = \begin{pmatrix}
\frac{\partial z_1}{\partial y_1} & \frac{\partial z_1}{\partial y_2} & \frac{\partial z_2}{\partial y_1} & \frac{\partial z_2}{\partial y_2} \\
\frac{\partial z_3}{\partial y_1} & \frac{\partial z_3}{\partial y_2} & \frac{\partial z_4}{\partial y_1} & \frac{\partial z_4}{\partial y_2}
\end{pmatrix} = \begin{pmatrix} 2y_1 & -1 & 2y_1 & 2 \\
-2 & 2y_2 & 2y_2 & 1
\end{pmatrix}.$$ 

Therefore,

$$\frac{\partial z}{\partial x} \frac{\partial y}{\partial z} = \begin{pmatrix} 2x_1 & 0 \\
-1 & 3 \\
0 & 2x_3
\end{pmatrix} \begin{pmatrix} 2y_1 & -1 & 2y_1 & 2 \\
-2 & 2y_2 & 2y_2 & 1
\end{pmatrix} = \begin{pmatrix} 4x_1y_1 & -2x_1 & 4x_1y_1 & 4x_1 \\
-2y_1 - 6 & 1 + 6y_2 & -2y_2 + 6y_2 & 1 \\
-4x_3 & 4x_3y_2 & 4x_3y_2 & 2x_3
\end{pmatrix}.$$
In Matlab

```
>> z1=y1^2-2*y2;
>> z2=y2^2-y1;
>> z3=y1^2+y2^2;
>> z4=2*y1+y2;
>> Jzx=jacobian([z1; z2; z3; z4],[x1 x2 x3])
Jzx =
[ 4*(x1^2-x2)*x1, -2*x1^2+2*x2-6, -4*x3]
[ -2*x1, 6*x3^2+18*x2+1, 4*(x3^2+3*x2)*x3]
[ 4*(x1^2-x2)*x1, -2*x1^2+20*x2+6*x3^2, 4*(x3^2+3*x2)*x3]
[ 4*x1, 1, 2*x3]
>> Jzx'
ans =
[ 4*(x1^2-x2)*x1, -2*x1, 4*(x1^2-x2)*x1, 4*x1]
[ -2*x1^2+2*x2-6, 6*x3^2+18*x2+1, -2*x1^2+20*x2+6*x3^2, 1]
[ -4*x3, 4*(x3^2+3*x2)*x3, 4*(x3^2+3*x2)*x3, 2*x3]
```
lately, i've just felt like nothing inside...

i can definitely identify with how you are feeling...

Association of Psychiatric Medicine

\[\begin{array}{cccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array}\]